Astrophysical Cosmology 4 2004/2005

Solution set 3

(1) The equation for a radial null geodesic in the Robertson-Walker metric is $dr = c\,dt/R(t)$. Using the relation between redshift and scale factor, $1+z \propto 1/R(t)$, plus Friedmann's equation, deduce the differential relation between comoving distance and redshift:

$$R_0\,dr = \frac{c}{H_0}\,\left[(1-\Omega)(1+z)^2 + \Omega_v + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2}\,dz.$$

Solution: Differentiating $1+z = R_0/R(t)$ wrt time we find $dz/dt = -R_0/R^2 dR/dt = -(1+z)H(z)$. Hence

$$dt = \tfrac{dz}{(1+z)H(z)}$$

which we shall use below. Substituting into $dr = c\,dt/R(t)$ we find

$$R_0 dr = \tfrac{c dz}{H(z)}.$$

We can find $H(z)$ from the Friedmann equation;

$$H^2 = \tfrac{8\pi G\rho}{3} - \tfrac{kc^2}{R^2},$$

by substituting kc^2/R_0^2 for $(H_0^2 - 8\pi G\rho_0/3)(1+z)^2$ and put $\rho(z) = \rho_0^m(1+z)^3 + \rho_0^v + \rho_0^r(1+z)^4$. Recall that $\Omega = 8\pi G\rho_0/(3H_0^2)$ and we get the Hubble parameter as a function of z;

$$H^2(z) = H_0^2[\Omega_m(1+z)^3 + \Omega_v + \Omega_r(1+z)^4 + (1-\Omega)(1+z)^2],$$

where $\Omega = \Omega_m + \Omega_v + \Omega_r$.

(2) Integrate this expression for the case of the $\Omega = 1$ Einstein–de Sitter universe. In this model, calculate the apparent angle subtended by a galaxy of proper diameter 30 kpc, as a function of redshift (recall $c/H_0 = 3000\,h^{-1}$ Mpc). Show that there is a critical redshift at which this angle has a minimum value.

Solution: EdS has $\Omega_m = \Omega = 1$ and $\Omega_v = \Omega_r = 0$, so

$$R_0 dr = \tfrac{c}{H_0}(1+z)^{-3/2} dz.$$

Integrating from us at $z = 0$ to the object at redshift z, the proper distance is

$$R_0 r = \tfrac{c}{H_0} \int_0^z dz\,(1+z)^{-3/2} = \tfrac{2c}{H_0}(1 - (1+z)^{-1/2}).$$

Note the limits on the integration (we are measuring the distance from us to a high-redshift object), and we have chosen minus signs so the distance from here to there is positive. As a check we can take the low redshift limit, $z \ll 1$, to get $R_0 r = cz/H_0$ (Hubble's Law). For $z \gg 1$ this converges to $R_0 r = 2c/H_0$, as we run up against the particle horizon (i.e. photons only have the age of the universe to travel a finite distance to us).

Using the small-angle formula, $d = \theta D_A(z)$, where $D_A(z) = R_0 r(z)/(1+z)$ is the angular diameter distance, for a fixed diameter, $d = 30$kpc, the apparent angle of an object subtended on the sky is

$$\theta(z) = \frac{d(1+z)}{\frac{2c}{H_0}(1-(1+z)^{-1/2})}.$$

As $c/H_0 = 3000h^{-1}$Mpc we find

$$\theta(z) = 1.03h((1+z)^{-1} - (1+z)^{-3/2})^{-1}\,\text{arcsec}.$$

This has a minimum, as can be seen by taking the low-z limit, $\theta(z) \propto 1/z$ and the high-z limit $\theta(z) \propto z$. So at low redshift the apparent angular size decreases with distance, as in a Euclidean universe, but at higher redshift it starts to increase again. This is because the matter in the universe is bending the light, so it's like looking through a gold-fish bowel. The minimum can be found by looking for the turning point $d\theta(z)/dz = 0$. So

$$\tfrac{d\theta}{dz} \propto \tfrac{d}{dz}((1+z)^{-1} - (1+z)^{-3/2})^{-1} = (-(1+z)^{-2} - 3/2(1+z)^{-5/2}) = 0$$

giving $(1+z)^{5/2-2} = 3/2$, so $1+z = (3/2)^2 = 9/4$. Hence the turning point is $z = 5/4$.

(3) From the relation between distance and redshift, deduce the differential relation between redshift and time. Integrate this for the case of the $\Omega = 1$ Einstein–de Sitter universe. A galaxy is observed at redshift 1.5 to contain stars that are 3.5 Gyr old: assuming $\Omega = 1$, deduce a limit on the Hubble constant (remember $H_0^{-1} = 9.78h^{-1}$ Gyr).

Solution: We've already got the first part of this from differentiating the redshift-scale factor relation; $dt = dz/(1+z)H(z)$. To find the time it's taken light to travel from redshift z to Earth, at $z = 0$, we need to integrate from $z = 0$ to z;

$$t(z) = \int_0^z \frac{dz}{(1+z)H(z)}.$$

But if we want to know the age of the universe (from $t = 0$) at redshift z we need to integrate from z to $z = \infty$ (remember $1 + z$ is the ratio of the size of the universe today to its size at redshift z, so for $R = 0$, $z = \infty$). Hence

$$t_{\rm age}(z) = \int_z^\infty \tfrac{dz}{(1+z)H(z)}.$$

In a EdS universe the age of the universe at z is

$$t_{\rm age}(z) = \tfrac{1}{H_0}\int_z^\infty \tfrac{dz}{(1+z)^{5/2}} = \tfrac{2}{3H_0}(1+z)^{-3/2}.$$

If we see a galaxy at $z = 1.5$ and can tell its age (e.g. from fitting stellar models to its spectral energy distribution) is $t = 3.5$Gyrs, then we know the universe must be at least this old, so we can solve for H_0 (note that $H_)^{-1} = 9.78h^{-1}$Gyrs). For $z = 1.5$ we find $t = 0.169/H_0$. or $t = 1.65h^{-1}$Gyrs. As $t > 3.5$Gyrs then $h < 0.47$.

(4) Show that the result of integrating dr/dz between ∞ and z is the comoving horizon length. For redshifts below about 10^4, the universe may be assumed to be matter dominated: deduce an expression for the horizon length as a function of z that is valid for $10^4 \gtrsim z \gtrsim 1$.

Solution: Integrating dr/dz from $z = \infty$ to z will give us the distance a photon has traveled from $z = \infty$, i.e. the big bang at $t = 0$ and $R = 0$, until the redshift z. As this is the furthest distance that anything can have traveled by that time in the evolution of the universe, this must equal the particle horizon at redshift z. In other words

$$r_H(z) = \int_0^t \tfrac{dr}{dt}dt = \int_\infty^z \tfrac{dr}{dz}dz.$$

If the universe is matter dominated then the horizon distance is

$$R_0 r_H(z) = \tfrac{c}{H_0}\int_\infty^z dz(1+z)^{-3/2}$$

(its ok to integrate to infinity, as it won't add much to the integral for $z > 10^4$), giving

$$R_0 d_H(z) = \tfrac{2c}{H_0}(1+z)^{-1/2}.$$

The microwave background was emitted at $z \simeq 1100$. Calculate the horizon size at this epoch. If $\Omega = 1$, what angle on the sky does this length subtend? Is it surprising that the microwave background radiation is uniform to 1 part in 1000?

Solution: At the redshift of the CMB, $z = 1100$, this gives $d_h = 181\,h^{-1}$Mpc (Note that this is the comoving size, i.e. the size it would be if we scaled it with the expansion to the present today). The angle this physical size subtends on the sky today can be found from question (2), but its quicker to notice that we are in the high-redshift regime, $z \gg 1$, so that $D_A(z \gg 1) \approx 2c/[H_0(1+z)]$, where $2c/H_0$ is the current horizon size, $r_H(0)$. Also note that the physical size of the horizon at z is $d_H(z) = R(z)r_H(z) \propto r_H(z)/(1+z)$.

Putting these together we find the angular size of the horizon at $z = 10^4$, for and EdS model, is

$$\theta(z \gg 1) = \frac{r_H(z)}{r_H(0)} = (1+z)^{-1/2},$$

which gives an angular size of $\theta(z = 1100) = 1.73$ degrees. Hence larger angular scales at the distance of the CMB where never in causal contact, making the fact that points separated by 180 degrees are the same temperature to within one part in 1000 rather strange!