Astrophysical Cosmology 4 2004/2005

Solution set 2

(1) Write down Friedmann's equation for the scale factor of the universe, $R(t)$. Assuming that the universe contains pressureless matter only, so that $\rho(t) = \rho_0(R/R_0)^{-3}$, rewrite the equation in terms of the variable η, where $d\eta = c\,dt/R(t)$.

Solution: The Friedmann equation is

$$\dot{R}^2 = \frac{8\pi G}{3}\,\rho R^2 - c^2 k.$$

If $\rho(t) = \rho_0(R/R_0)^{-3}$ then

$$\dot{R}^2 = \frac{8\pi G}{3}\,\rho_0 R_0^3\, R^{-1} - c^2 k.$$

Since $d\eta = c\,dt/R(t)$ then $d/dt = c/R d/d\eta$. Substituting in we find

$$R'^2 = \frac{8\pi G \rho_0 R_0^3}{3c^2}\, R - kR^2.$$

If we let $R_* = 4\pi G\rho_0 R_0^3/3c^2$ then

$$(R/R_*)'^2 = 2(R/R_*) - k(R/R_*)^2,$$

which is the form required.

(2) Hence verify that $R(\eta) = R_*(1-\cos\eta)$ and $R_*(\cosh\eta - 1)$ are the solutions for respectively positive and negative space curvature, where $R_* = (c/H_0)\,\Omega\,|1-\Omega|^{-3/2}/2$ (remember $R_0 = (c/H_0)\,|1-\Omega|^{-1/2}$). Use $dt = R\,d\eta/c$ to deduce $t(\eta)$.

Solution:

Let $k = +1$, so $R(\eta) = R_*(1-\cos\eta)$. Then

$$(R/R_*)'^2 = \sin^2\eta$$

and

$$\begin{aligned} 2(R/R_*) - (R/R_*)^2 &= 2(1-\cos\eta) - (1-\cos\eta)^2 \\ &= 1-\cos^2\eta \\ &= \sin^2\eta \\ &= (R/R_*)'^2 \end{aligned}$$

This proves the spatially close case. The open case follows by letting $R \to iR$.

The cosmic time is given by

$$t = \int_0^t dt = \int_0^\eta R(\eta)/c = \frac{1}{c} R_*(\eta - \sin\eta)$$

(3) If $H_0 = 70\,\mathrm{km\,s^{-1}Mpc^{-1}}$ and we inhabit a matter-dominated universe with $\Omega = 1.1$, then

(a) What is the current age of the universe?
(solve $R(\eta)$ for η_0 and use $H_0^{-1} = 9.78\,h^{-1}\mathrm{Gyr}$)

Solution:

The age of the universe is given by

$$t_0 = R_*(\eta_0 - \sin\eta_0)/c,$$

and its scale factor is

$$R(\eta_0) = R_*(1-\cos\eta_0).$$

Inverting this we find $\eta_0 = \cos^{-1}(1 - R_0/R_*)$. Since $R_0 = (c/H_0)\,|1-\Omega|^{-1/2}$ and $R_* = (c/H_0)\,\Omega\,|1-\Omega|^{-3/2}/2$, then after a bit of algebra $1 - R_0/R_* = 2\Omega^{-1} - 1$. Hence $\eta_0 = \cos^{-1}(2\Omega^{-1} - 1) = 0.613$, and so

$$t_0 = \frac{\Omega}{2H_0|1-\Omega|^{3/2}}(\eta_0 - \sin\eta_0) \approx 0.654 H_0^{-1}.$$

Note that this is less than the naive extrapolation of $t_0 = H_0^{-1}$, and slightly lower than the Einstein-de Sitter value of $t_0 = 2/3H_0^{-1}$. With $H_0 = 70\mathrm{kms}^{-1}\mathrm{Mpc}^{-1} = 1/9.78h^{-1}\mathrm{Gyr}$, then we find an age to this model of $t_0 = 9.136\mathrm{Gyr}$.

(b) How much larger will the universe be when it ceases expanding?

Solution: This universe will stop expanding at the point of turnaround, when the expansion stops so that $\dot{R} \propto R' = 0$. Afterwards it will recollapse to a big crunch. Since $\dot{R} = R_*\sin\eta = 0$, we find $\eta = \pi$ at turnaround. The scale factor is then $R(\eta=\pi) = R_*(1-\cos\pi) = 2R_*$. This is bigger than the current scale factor by a factor $R(\eta=\pi)/R_0 = 2R_*/R_*(1-\cos\eta_0) = 11.0$.

(c) When will this happen?

Solution: Turnaround happens at a time $t_{\max} = R_*(\pi - \sin\pi)/c$, or a factor $t_{\max}/t_0 = (\pi - \sin\pi)/(\eta_0 - \sin\eta_0) = 83.6$ older, making $t_{\max} = 763$ Gyrs.

(d) What will $H(t)$ and $\Omega(t)$ be at this time?

Solution: Since $\dot{R} = 0$ at maximum expansion, $H = \dot{R}/R = 0$. As $\Omega(t) \propto 1/H(t)^2$, we see that Ω is infinite at turnaround.

(e) At what time will the big crunch happen?

Solution: From the symmetry of the expansion and collapse of the model, the big crunch will happen at a time $2 \times t_{\text{max}}$.

(4) Suppose instead that we inhabit a universe with a significant vacuum energy. If the matter and vacuum densities are $\Omega_m = 0.1$ and $\Omega_v = 1.5$, show that there could not have been a big bang. What is the highest redshift we could expect to see? (This will require some numerical experimentation with your calculator).

Solution: As the vacuum energy increases, the big-bang gets pushed back further and further. Eventually, when its large enough there is no singularity. Instead the universe starts off collapsing from infinitely large in the infinite past, reaches a finite radius, 'bounces', and starts to expand. Hence the signature of no initial singularity is the presence of a point to inflection, or stationary point when $\dot{R} = 0 = H$, in the past. As this is given by Friedmann's equation written in the form

$$H(t)^2 = H_0^2[\Omega_m(1+z)^3 + \Omega_v + (1 - \Omega_m - \Omega_v)(1+z)^2].$$

So we are looking for when the term in square brackets is zero. Plugging some values into a calculator shows that this happens at about $z \approx 0.9$. More distant objects will still be in the collapsing part of the universe and so will be blue-shifted.