Astrophysical Cosmology 4 2004/2005

Solutions 1

(1) The Robertson-Walker metric can be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left(dr^2 + S_k^2(r)\, d\psi^2\right),$$

where $S_k(r) = \sin r$ ($k = +1$), r ($k = 0$), or $\sinh r$ ($k = -1$) and $d\psi$ is the angular separation between the two events under consideration.

(a) Show that, in the $k = +1$ case, the spatial part of this 3D metric is the same as that for a 2D space that forms the surface of a sphere.

Solution: If $d\psi$ is a general angular separation on the sky, then

$$d\psi^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the line element (an infinitesimal Pythagoras Theorem). This is the same form as the spatial part of the RW metric if we replace θ by r (a dimensionless distance from the pole of the sphere), and replace $d\phi$, an infinitesimal angular distance on a circle of constant θ around the unit sphere, with $d\psi$, and infinitesimal angular distance between two points on the surface of a 2-sphere.

(b) The proper time interval $d\tau$ is not in general the same as the time interval measured by an observer at the origin. By considering a pair of events with $dr = d\psi = 0$, explain why not.

Solution: Lets say we are at the origin and $dr = d\psi = 0$. Then $c^2 d\tau^2 = c^2 dt^2$, i.e., the proper time is the same as the time measured by the fundamental observer at the origin. But the apparent rate of time of distant galaxies will appear slower, as they are moving away and so will be time-dilated, i.e. $c^2 d\tau^2 = c^2 dt^2 - R^2 dr^2$.

(c) The element of radial proper distance is $R(t)\, dr$. How does this change with time? By considering events close to $r = 0$, deduce Hubble's law, and show that $H = \dot{R}/R$.

Solution: Lets call the element of proper (i.e., physical) distance $dl = R(t)dr$. Remembering that the comoving distance, r, is time-independent, the rate of change of the proper distance with time is $v = (dl)^{\cdot} = \dot{R}dr = (\dot{R}/R)dl = H dl$, yielding Hubble's law.

(d) Show that the curved $k = \pm 1$ cases are in practice indistinguishable from the flat $k = 0$ case provided R is large enough.

Solution: The Robertson-Walker metric can be written as

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\, \left(dr^2 + S_k^2(r)\, d\psi^2\right),$$

where $S_k(r) = \sin r$ $(k = +1)$, r $(k = 0)$, or $\sinh r$ $(k = -1)$. Expanding $S_k(r) \simeq r - kr^3/6$, to third order, and substituting the comoving distance r for the proper distance $l = Rr$, we can write the RW metric as

$$c^2 d\tau^2 = c^2 dt^2 - \left(dl^2 + [l^2 - k(l^4/R^2)/3]d\psi^2\right).$$

Hence if we let $R \to \infty$ we can see the metric turns into the flat Minkowski metric of special relativity. The condition for this to happen is $R \gg l$, i.e. if the scale factor is very large, or if proper distances are small in comparison.

(2) Using the Robertson-Walker metric, show that the comoving separation between us and an object seen at redshift z is $r = \int c\, dt/R(t)$. Since r is independent of time, argue that the redshift is $1 + z = R(\text{now})/R(\text{emit})$.

Solution: We see galaxies by the photons they emit. Photons, being massless, travel along null geodesics (i.e. they see zero proper time), $c^2 d\tau^2 = 0 = c^2 dt^2 - R^2(t)dr^2$. Solving for $dr = dt/R(t)$ and integrating we find

$$r(t) = c \int_0^t \frac{dt'}{R(t')}$$

is the proper distance the photon has travelled. We can use this to find the redshift formula for an expanding universe. Imagine a photon is emitted from a galaxy at time $t = t_0$, and arrives at an observer at time t_1, having travelled a comoving distance r. A short time later another photon is emitted at $t = t_0 + \delta t_0$ and is observed at a time $t = t_1 + \delta t_1$. While the physical distances between galaxies has increased due to the expansion, the comoving distance is still r. Hence

$$r = c \int_{t_0}^{t_1} \frac{dt'}{R(t')} = c \int_{t_0+\delta t_0}^{t_1+\delta t_1} \frac{dt'}{R(t')}$$

The second integral can be expanded as $\int_{t_0+\delta t_0}^{t_1+\delta t_1} = \int_{t_0}^{t_1} - \int_{t_0}^{t_0+\delta t_0} + \int_{t_1}^{t_1+\delta t_1}$. Cancelling the $\int_{t_0}^{t_1}$ integrals we find

$$\int_{t_0}^{t_0+\delta t_0} \frac{dt'}{R(t')} = \int_{t_1}^{t_1+\delta t_1} \frac{dt'}{R(t')}.$$

For small $\delta t_0, \delta t_1$ we find

$$\frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)}.$$

We can always choose the time interval so that $\lambda_0 \propto \delta t_0$, hence

$$\frac{\lambda_0}{R(t_0)} = \frac{\lambda_1}{R(t_1)}.$$

Since $\nu \propto \lambda$ and $\nu_{\rm emit}/\nu_{\rm obs} = 1 + z$, and letting $t_0 = t_{\rm emit}$ and the time of observing be $t_1 = t_{\rm now}$ then $1 + z = R(t_{\rm now})/R(t_{\rm emit})$.

(3) Derive the same relation by considering the infinitesimal Doppler shift caused when a photon travels a distance d, thus encountering an observer with relative velocity $\delta v = Hd$ (use $d = c\,\delta t$ and remember $H = \dot{R}/R$). Use the same approach to show that the 'peculiar' momentum of any particle decays $\propto 1/R$ (use a Lorentz transformation of the particle 4-momentum to get the change in momentum caused by δv).

Solution: The fractional change in frequency of light caused by a small Doppler shift is $\delta\nu/\nu = +\delta v/c$. The Doppler shift is seen by the observer, who sees the emitting galaxy moving away with velocity $-\delta v$. Hence

$$\delta\nu/\nu = -\delta v/c = -Hd/c = -H\delta t = -\dot{R}/R\delta t = -\delta R/R.$$

Hence $\nu \propto 1/R$.

The 4-momentum of a particle is $p^\mu = (E/c, p_x, p_y, p_z)$, measured by a nearby fundamental observer. If we perform a Lorentz transformation to another frame, i.e. as seen from a galaxy a distance d away moving with a relative Hubble velocity of $\delta v = Hd$, $p'_x = \gamma(p_x - \delta v E/c^2)$. Hence to first order in δv the 'peculiar' momentum, i.e. the momentum of a particle with respect to the Hubble expansion measured by the fundamental observers, is $\delta p \simeq -\delta v E/c^2$ (dropping un-important subscripts). Since $E = mc^2$ then $\delta p \simeq -m\delta v$ (i.e. the Newtonian limit where $v \ll c$). As $\delta v = Hd$ is just the Hubble velocity $\delta p = -Hmd$. The distance $d = \delta t p/m$ so $\delta p = -Hp\,\delta t$ or $\delta p/p = -H\delta t = -\delta R/R$. Hence $p \propto 1/R$.