FoMP: Vectors, Tensors and Fields

Example problems with solutions

1 *Given that the vector $\underline{a}$ has components $(1, 2, 3)$ in a orthonormal, right-handed basis:–*

(i) *construct a* unit *vector $\hat{\underline{a}}$ parallel to $\underline{a}$;*

(ii) *construct a vector $\underline{b}$ orthogonal to $\underline{a}$, with no z-component (i.e. $b_3 = 0$) ;*

(iii) *construct a unit vector $\hat{\underline{b}}$ parallel to $\underline{b}$;*

(iv) *construct a vector $\underline{c}$ orthogonal to both $\underline{a}$ and $\underline{b}$ by requiring that $\underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} = 0$;*

(v) *construct a unit vector $\hat{\underline{c}}$ orthogonal to both $\hat{\underline{a}}$ and $\hat{\underline{b}}$, such that $\hat{\underline{a}}$, $\hat{\underline{b}}$ and $\hat{\underline{c}}$ form a right-handed triad ;*

(vi) *Verify that $\hat{\underline{c}} = \hat{\underline{a}} \times \hat{\underline{b}}$.*

(i) Construct $\hat{\underline{a}} = \underline{a}/a$, where a is the magnitude of $\underline{a}$. Now $a^2 = \underline{a} \cdot \underline{a} = 14$, so that in principle, we could take $a = \pm 1/\sqrt{14}$ but we choose the positive root so that $\hat{\underline{a}}$ is parallel, rather than anti-parallel to $\underline{a}$. The desired unit vector is thus

$$\hat{\underline{a}} = \frac{1}{\sqrt{14}}\,(1, 2, 3)$$

(ii) The vector $\underline{b}$ is of the form $(b_1, b_2, 0)$ and orthogonal to $\underline{a}$ so that $\underline{b} \cdot \underline{a} = 0$, giving $b_1 + 2b_2 = 0$. If we choose $b_2 = 1$, say, then the desired vector is

$$\underline{b} = (-2, 1, 0)$$

Clearly an arbitrary scalar multiple of this would also satisfy the orthogonality requirement.

(iii) With the above choice for $\underline{b}$, we have $b^2 = \underline{b} \cdot \underline{b} = 5$. Again we take the positive square root to yield

$$\hat{\underline{b}} = \frac{1}{\sqrt{5}}\,(-2, 1, 0)$$

(iv) Writing $\underline{c} = (c_1, c_2, c_3)$ and demanding that $\underline{c} \cdot \underline{a} = 0$ tells us that $c_1 + 2c_2 + 3c_3 = 0$. Similarly, $\underline{c} \cdot \underline{b} = 0$ implies that $-2c_1 + c_2 = 0$. Thus $5c_1 = -3c_3$ and $5c_2 = -6c_3$. If we choose $c_1 = 3$ we obtain

$$\underline{c} = (3, 6, -5)$$

(v) With this choice of $\underline{c}$, we see that $c^2 = \underline{c}\cdot\underline{c} = 70$ so that $c = \pm 1/\sqrt{70}$. The choice of sign is dictated by the requirement that $\hat{\underline{a}}, \hat{\underline{b}}, \hat{\underline{c}}$ form a right-handed triad, so we require that $(\hat{\underline{a}}, \hat{\underline{b}}, \hat{\underline{c}}) = 1$:

$$(\hat{\underline{a}}, \hat{\underline{b}}, \hat{\underline{c}}) = \pm \begin{vmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{3}{\sqrt{70}} & \frac{6}{\sqrt{70}} & \frac{-5}{\sqrt{70}} \end{vmatrix} = \pm\frac{1}{70}\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & 6 & -5 \end{vmatrix} = \pm(-1)$$

Thus we must take the - sign, giving $\quad \hat{\underline{c}} = \frac{1}{\sqrt{70}}(-3, -6, 5)$.

(vi) From the definition of the vector product

$$\hat{\underline{a}} \times \hat{\underline{b}} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{vmatrix} = -\underline{e}_1\frac{3}{\sqrt{70}} - \underline{e}_2\frac{6}{\sqrt{70}} + \underline{e}_3\frac{5}{\sqrt{70}} = \hat{\underline{c}}$$

2 *(i)* *The velocity of a point particle, rotating with angular velocity vector $\underline{\omega}$ through the origin, is $\underline{v} = \underline{\omega} \times \underline{r}$. The angular momentum vector of such a particle is $\underline{L} = \underline{r} \times m\underline{v}$ where m is the mass.*
Show that L_ω, the component of the angular momentum vector along the axis of rotation, is given by

$$L_\omega = m\omega\left[r^2 - |\underline{r}\cdot\hat{\underline{\omega}}|^2\right]$$

What is the geometrical meaning of the term in the square bracket?

(ii) *The kinetic energy of such a particle is given by $K = m(\underline{v}\cdot\underline{v})/2$.*
Show that

$$K = \frac{mw^2}{2}\left[r^2 - |\underline{r}\cdot\hat{\underline{\omega}}|^2\right]$$

The definition of the required component is $L_\omega = \underline{L}\cdot\hat{\underline{\omega}}$

In full, the angular momentum is $\underline{L} = m\underline{r} \times (\underline{\omega} \times \underline{r}) = m\,\omega\,\underline{r} \times (\hat{\underline{\omega}} \times \underline{r})$.

Expanding the triple vector product using BAC-CAB, this gives $\underline{L} = m\omega\left[\hat{\underline{\omega}}r^2 - \underline{r}(\underline{r}\cdot\hat{\underline{\omega}})\right]$. Taking $\underline{L}\cdot\hat{\underline{\omega}}$ now gives the required result.

The term in square brackets is $r^2(1 - |\hat{\underline{r}}\cdot\hat{\underline{\omega}}|^2) = (r\sin\theta)^2$, where θ is the angle between $\hat{\underline{r}}$ and $\hat{\underline{\omega}}$. $R = r\sin\theta$ is the radius of the circle around which the particle moves. From this 2D point of view, we would clearly expect $L_\omega = m\omega R^2$.

The definition of K involves $\underline{v}\cdot\underline{v} = (\underline{\omega}\times\underline{r})\cdot(\underline{\omega}\times\underline{r})$. Using the cyclic property of the triple scalar product, this becomes $(\underline{r}\times[\underline{\omega}\times\underline{r}])\cdot\underline{\omega}$.

Expanding $\underline{r} \times [\underline{\omega} \times \underline{r}]$ using BAC-CAB, we get $\underline{\omega} r^2 - \underline{r}(\underline{\omega} \cdot \underline{r})$.

Taking the dot product with $\underline{\omega}$, we then get $\underline{v} \cdot \underline{v} = \omega^2 r^2 - (\underline{r} \cdot \underline{\omega})^2 = \omega^2 (r^2 - (\underline{r} \cdot \underline{\hat{\omega}})^2$.

3 *The two sets of basis vectors $\{\underline{e}_i\}$ and $\{\underline{e}_i{}'\}$ are both right-handed orthonormal triads such that $\underline{e}_1{}'$ is in the direction of $(\underline{e}_1 - \underline{e}_3)$ and $\underline{e}_2{}'$ is in the direction of $(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$.*

(i) *Construct the correctly normalised basis vectors $\underline{e}_1{}', \underline{e}_2{}', \underline{e}_3{}'$.*

(ii) *Write down the transformation matrix $\underline{\underline{\lambda}}$ from the basis $\{\underline{e}_i\}$ to the basis $\{\underline{e}_i{}'\}$.*

(iii) *Express the two vectors $\underline{A} = 2\underline{e}_1 + \underline{e}_3$ and $\underline{B} = \underline{e}_1 - \underline{e}_2 - \underline{e}_3$ in the basis $\{\underline{e}_i{}'\}$.*

(iv) *Verify that the scalar product of these two vectors is an invariant of the transformation.*

(v) *Show that the components of the vector product, evaluated in the two bases, are related as follows:*

$$(\underline{A} \times \underline{B})_i{}' = \lambda_{ij} \, (\underline{A} \times \underline{B})_j$$

(i) $\underline{e}_1{}'$ is a unit vector in the direction of $\underline{e}_1 + \underline{e}_3$ and so

$$\underline{e}_1{}' \;=\; \frac{1}{\sqrt{2}} \, (\underline{e}_1 - \underline{e}_3)$$

$\underline{e}_2{}'$ is a unit vector in the direction of $\underline{e}_1 + \underline{e}_2 + \underline{e}_3$ and so

$$\underline{e}_2{}' \;=\; \frac{1}{\sqrt{3}} \, (\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$$

$\underline{e}_3{}' = \underline{e}_1{}' \times \underline{e}_2{}'$ since the basis $\{\underline{e}_i{}'\}$ is right-handed

$$\begin{aligned} \underline{e}_3{}' &= \frac{1}{\sqrt{6}} \, (\underline{e}_1 - \underline{e}_3) \times (\underline{e}_1 + \underline{e}_2 + \underline{e}_3) = \frac{1}{\sqrt{6}} \, (\underline{e}_3 - \underline{e}_2 - \underline{e}_2 + \underline{e}_1) \\ &= \frac{1}{\sqrt{6}} \, (\underline{e}_1 - 2\underline{e}_2 + \underline{e}_3) \end{aligned}$$

(ii) The elements of $\underline{\underline{\lambda}}$ are given by $\lambda_{ij} = \underline{e}_i{}' \cdot \underline{e}_j$ which can be read off directly from (i)

$$\underline{\underline{\lambda}} \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

(iii) $A_i' = \lambda_{ij} A_j$ so that, given $A_1 = 2$, $A_3 = 1$ and $A_2 = 0$ we find that $A_1' = 1/\sqrt{2}$, $A_2' = 3/\sqrt{3}$ and $A_3' = 3/\sqrt{6}$, giving

$$\underline{A} \;=\; \frac{1}{\sqrt{2}} \, \underline{e}_1{}' + \frac{3}{\sqrt{3}} \, \underline{e}_2{}' + \frac{3}{\sqrt{6}} \, \underline{e}_3{}'$$

Similarly, $B'_i = \lambda_{ij} B_j$ and $B_1 = 2/\sqrt{2}$, $B_2 = -1/\sqrt{3}$ and $B_3 = 2/\sqrt{6}$, yielding

$$\underline{B} \;=\; \frac{2}{\sqrt{2}}\,\underline{e}_1{}' - \frac{1}{\sqrt{3}}\,\underline{e}_2{}' + \frac{2}{\sqrt{6}}\,\underline{e}_3{}'$$

(iv) The scalar product is $\underline{A} \cdot \underline{B} = A_i B_i = A'_i B'_i$. To check this we calculate

$$\begin{aligned} A_i B_i &= 2\,1 + 0 - 1\,1 = 1 \quad \text{and} \\ A'_i B'_i &= \frac{1}{\sqrt{2}}\sqrt{2} - \sqrt{3}\frac{1}{\sqrt{3}} + \sqrt{\frac{3}{2}}\sqrt{\frac{2}{3}} = 1 \end{aligned}$$

(v) Assuming the formula $(\underline{A} \times \underline{B})_i = \epsilon_{ijk} A_j B_k$

$$\begin{aligned} (\underline{A} \times \underline{B})_1 &= A_2 B_3 - A_3 B_2 &=& \;\; 1 \\ (\underline{A} \times \underline{B})_2 &= A_3 B_1 - A_1 B_3 &=& \;\; 3 \\ (\underline{A} \times \underline{B})_3 &= A_1 B_2 - A_2 B_1 &=& \; -2 \end{aligned}$$

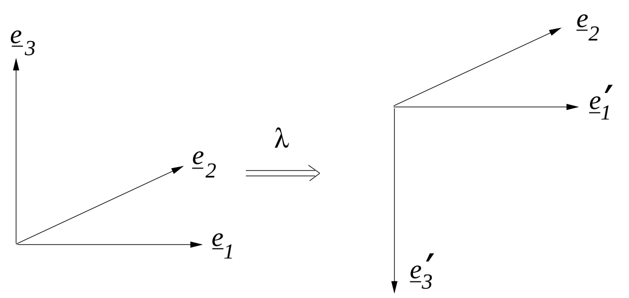
$$\begin{aligned} (\underline{A} \times \underline{B})'_1 &= A'_2 B'_3 - A'_3 B'_2 &=& \;\; 3/\sqrt{2} \\ (\underline{A} \times \underline{B})'_2 &= A'_3 B'_1 - A'_1 B'_3 &=& \;\; 2/\sqrt{3} \\ (\underline{A} \times \underline{B})'_3 &= A'_1 B'_2 - A'_2 B'_1 &=& \; -7/\sqrt{6} \end{aligned}$$

so we can check the relations between components in the two bases:

$$\begin{aligned} \lambda_{1j}(\underline{A} \times \underline{B})_j &= \;\; 3/\sqrt{2} &=& \; (\underline{A} \times \underline{B})'_1 \\ \lambda_{2j}(\underline{A} \times \underline{B})_j &= \;\; 2/\sqrt{3} &=& \; (\underline{A} \times \underline{B})'_2 \\ \lambda_{3j}(\underline{A} \times \underline{B})_j &= -7/\sqrt{6} &=& \; (\underline{A} \times \underline{B})'_3 \end{aligned}$$

4 *Construct the elements λ_{ij} of the transformation matrix and calculate its determinant for (i) a reflection of basis in the $\underline{e}_1$–$\underline{e}_2$ plane; (ii) a rotation of basis through an angle θ about the $\underline{e}_3$ axis where a positive rotation is taken to be in r.h. screw direction; (iii) a rotation of basis through an angle θ about the $\underline{e}_1$ axis where a positive rotation is taken to be in the r.h. screw direction. Show that the matrices in (ii) and (iii) do not commute. Finally, consider the case of small rotations and work to first order in angles, so that* $\cos\theta \to 1$ *and* $\sin\theta \to \theta$, *and show that the rotations now commute. Thus argue that angular velocity can be represented by a vector, even though rotations cannot.*

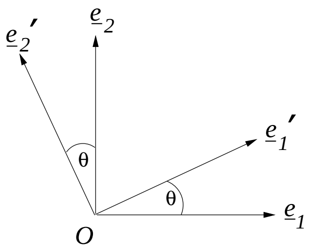
(i) Reflection in the $\underline{e}_1$–$\underline{e}_2$ plane:



$$\left.\begin{aligned} \underline{e}_1{}' &= \underline{e}_1 \\ \underline{e}_2{}' &= \underline{e}_2 \\ \underline{e}_3{}' &= -\underline{e}_3 \end{aligned}\right\} \Rightarrow \underline{\underline{\lambda}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det \lambda = -1$$

(ii) Rotation about $\underline{e}_3$:



$$
\begin{aligned}
\underline{e}_1{}' \cdot \underline{e}_1 &= \cos\theta \\
\underline{e}_1{}' \cdot \underline{e}_2 &= \sin\theta \\
\underline{e}_1{}' \cdot \underline{e}_3 &= 0 \\
\underline{e}_2{}' \cdot \underline{e}_1 &= -\sin\theta \\
\underline{e}_2{}' \cdot \underline{e}_2 &= \cos\theta \\
\underline{e}_2{}' \cdot \underline{e}_3 &= 0
\end{aligned}
$$

where we have chosen a r.h. basis and assumed the r.h. screw rule. Thus

$$\underline{\underline{\lambda}} = \underline{\underline{\lambda}}_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \det\lambda = \cos^2\theta + \sin^2\theta = 1$$

(iii) Rotation about $\underline{e}_1$: proceed similarly to get

$$\underline{\underline{\lambda}} = \underline{\underline{\lambda}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \det\lambda = \cos^2\theta + \sin^2\theta = 1$$

We now need to compute the product of these matrices. It is quicker to use the shorthand $C = \cos\theta$, $S = \sin\theta$:

$$\underline{\underline{\lambda}}_1\, \underline{\underline{\lambda}}_2 = \begin{pmatrix} C & CS & S^2 \\ -S & C^2 & SC \\ 0 & -S & C \end{pmatrix} \quad \underline{\underline{\lambda}}_2\, \underline{\underline{\lambda}}_1 = \begin{pmatrix} C & S & 0 \\ -SC & C^2 & S \\ S^2 & -SC & C \end{pmatrix}.$$

These two matrices differ. But if we take the small-angle limit and replace C by 1 and S by θ and S^2 by 0, then things do commute:

$$\underline{\underline{\lambda}}_1\, \underline{\underline{\lambda}}_2 = \underline{\underline{\lambda}}_2\, \underline{\underline{\lambda}}_1 = \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & \theta \\ 0 & -\theta & 1 \end{pmatrix}.$$

If we regard the rotation angle as $\theta = \omega\, dt$, this show that the rotations from two angular velocities will combine linearly.

5 *A certain type of anisotropically conducting crystal allows current to flow in one direction only, along which there is a linear response of current to applied voltage.. Suppose that this crystal is orientated so that current can flow only along the x axis:*

(i) Show that this statement can be written as a tensor relation between the applied electric field vector, $\underline{E}$ and the current density vector, $\underline{J}$, and write down the components of the conductivity tensor in this basis.

(ii) The system is now viewed using a system of coordinates rotated anticlockwise by an angle θ about the z axis. Give the transformation matrix $\underline{\underline{\lambda}}$ that relates vectors in the old and new coordinate systems.

(iii) Derive the general transformation law that must be satisfied by the components of a conductivity tensor in order to preserve the correct relation between electric field and current density in the new coordinate system, and apply it to deduce the transformed conductivity tensor in the present example.

(iv) Consider in particular rotations of $\pi/2$, π and 2π and comment on whether your mathematical results make physical sense.

(i) In the x direction, we have the usual Ohm's law: $J_x = g\, E_x$, where g is a conductivity measure. J_x is independent of E_y and E_z and $J_y = J_z = 0$. This can be written in tensor form $J_i = G_{ij}\, E_j$, where all components of G_{ij} are zero except $G_{11} = g$.

(ii) This is as usual from the notes:

$$\underline{\underline{\lambda}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) Changing basis cannot alter the form of a physical law. Therefore, if our relation reads $J_i = G_{ij} E_j$, we must have

$$J'_i = G'_{ij} E'_j \quad \text{where} \quad J'_i = \lambda_{ij} J_j \quad \text{and} \quad E'_j = \lambda_{jk} E_k.$$

Rewriting the original relation, we have $\lambda^{-1}_{ik} J'_k = G_{ij} \lambda^{-1}_{j\ell} E'_\ell$. Now multiply on the left by λ_{mi} and use the fact that $\lambda_{mi}\lambda^{-1}_{ik} = \delta_{mk}$. this gives

$$J'_m = \left(\lambda_{mi}\, G_{ij}\, \lambda^{-1}_{j\ell}\right)\, E'_\ell$$

and using the orthogonal property of the transformation matrix, we therefore identify the transformed components of the tensor:

$$G'_{m\ell} = \lambda_{mi}\, G_{ij}\, (\lambda^T)_{j\ell},$$

which in matrix notation is

$$\underline{\underline{G'}} = \underline{\underline{\lambda}}\, \underline{\underline{G}}\, \underline{\underline{\lambda}}^T.$$

Carrying out the matrix multiplication,

$$\begin{aligned} \underline{\underline{G'}} &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= g \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

(iv) Thus $\theta = \pi$ gives the same result as $\theta = 0$ or 2π. This makes sense: current runs in the x direction only, but can flow equally well in either direction. The case $\theta = \pi/2$ has the effect of swapping the x and y axes and indeed G_{ij} in that case says just $J_y = gE_y$.

6 *Three equal masses are placed at the origin, at $\underline{r} = 2a\underline{e}_1$ and at $\underline{r} = a\underline{e}_1 + \sqrt{3}a\underline{e}_2$.*

(i) *Calculate the centre of mass position vector, $\underline{R}$, and the position vectors, $\underline{s}^\alpha$, of the particles relative to the centre of mass.*

(ii) *Construct the inertia tensor* relative to the centre of mass.

(iii) *Use the parallel axes theorem to construct the inertia tensor relative to the origin*

(i) The position vector of the centre of mass is

$$\underline{R} = \frac{1}{M}\sum_\alpha m^\alpha \underline{r}^\alpha \quad \text{where} \quad M = \sum_\alpha m^\alpha = 3m$$

Thus

$$\underline{R} = \frac{1}{3m}\left[m(2a\underline{e}_1) + m(a\underline{e}_1 + \sqrt{3}a\underline{e}_2)\right] = \left(a\underline{e}_1 + \frac{a}{\sqrt{3}}\underline{e}_2\right)$$

Position vectors relative to the centre of mass are $\underline{s}^\alpha = \underline{r}^\alpha - \underline{R}$ thus

$$\underline{s}^{(1)} = \left(-a\underline{e}_1 - \frac{a}{\sqrt{3}}\underline{e}_2\right), \; \underline{s}^{(2)} = \left(a\underline{e}_1 - \frac{a}{\sqrt{3}}\underline{e}_2\right), \; \underline{s}^{(3)} = \left(\frac{2a}{\sqrt{3}}\underline{e}_2\right)$$

(ii) The inertia tensor relative to the centre of mass is

$$I_{ij}(G) = \sum_\alpha m^\alpha \left\{(\underline{s}^\alpha \cdot \underline{s}^\alpha)\,\delta_{ij} - s_i^\alpha s_j^\alpha\right\}$$

We can compute as follows:

$$\begin{aligned}
\sum_\alpha m^\alpha(s_1^\alpha s_1^\alpha) &= ma^2(1+1+0) = 2ma^2\\
\sum_\alpha m^\alpha(s_2^\alpha s_2^\alpha) &= ma^2(1/3+1/3+4/3) = 2ma^2\\
\sum_\alpha m^\alpha(s_3^\alpha s_3^\alpha) &= 0\\
\sum_\alpha m^\alpha(s_1^\alpha s_2^\alpha) &= \sum_\alpha m^\alpha(s_2^\alpha s_3^\alpha) = \sum_\alpha m^\alpha(s_3^\alpha s_1^\alpha) = 0\\
\sum_\alpha m^\alpha(\underline{s}^\alpha \cdot \underline{s}^\alpha) &= ma^2(4/3+4/3+4/3) = 4ma^2
\end{aligned}$$

Thus

$$\underline{\underline{I}}(G) = 4ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} = 2ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$$

An equally acceptable method is to calculate the inertia tensors for the individual masses and add them together, as in lectures.

(iii) The Parallel Axes Theorem states that

$$I_{ij}(O) - I_{ij}(G) = M\left\{(\underline{R}\cdot\underline{R})\,\delta_{ij} - R_iR_j\right\}$$

Now $M = 3m$, $\underline{R}\cdot\underline{R} = 4a^2/3$ and $R_1 = a$, $R_2 = a/\sqrt{3}$ and $R_3 = 0$. Thus

$$\begin{aligned}\underline{\underline{I}}(O) &= 2ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix} + 3m\times\frac{4a^2}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - ma^2\begin{pmatrix}3&\sqrt{3}&0\\\sqrt{3}&1&0\\0&0&0\end{pmatrix}\\ &= ma^2\begin{pmatrix}3&-\sqrt{3}&0\\-\sqrt{3}&5&0\\0&0&8\end{pmatrix}\end{aligned}$$

7 *With reference to a given Cartesian basis a second-rank, symmetric tensor $\underline{\underline{T}}$ has components*

$$\begin{pmatrix}4&1&1\\1&4&-1\\1&-1&2\end{pmatrix}.$$

Calculate its eigenvalues and eigenvectors, and verify that the latter are orthogonal. Construct a transformation matrix $\underline{\underline{\lambda}}$ whose rows *are the* **normalised** *eigenvectors of the tensor T. Verify that $\underline{\underline{\lambda}}\,\underline{\underline{\lambda}}^T = \underline{\underline{\delta}}$ (where $\underline{\underline{\delta}}$ denotes the identity matrix) and that $\underline{\underline{\lambda}}\,\underline{\underline{T}}\,\underline{\underline{\lambda}}^T$ is diagonal, with its diagonal components the eigenvalues of $\underline{\underline{T}}$.*

The characteristic equation is

$$\begin{vmatrix}4-t&1&1\\1&4-t&-1\\1&-1&2-t\end{vmatrix} = (4-t)[(4-t)(2-t)-1] - [(2-t)+1] - [-1-4+t] = 0$$

Thus

$$(4-t)[5-6t+t^2] = 0 \quad\Rightarrow\quad (1-t)(4-t)(5-t) = 0$$

Thus the eigenvalues are 1, 4 and 5.

For $t = t^{(1)} = 1$ we denote the corresponding eigenvector by $\underline{n}^{(1)}$ and the equations for the components of $\underline{n}^{(1)}$ are (dropping the label (1))

$$\begin{pmatrix}3&1&1\\1&3&-1\\1&-1&1\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad\Rightarrow\quad n_1 = -n_2,\;\; n_3 = 2n_2$$

Thus a normalised eigenvector is: $\hat{\underline{n}}^{(1)} = \dfrac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}$ For $t = t^{(2)} = 4$

$$\begin{pmatrix}0&1&1\\1&0&-1\\1&-1&-2\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad\Rightarrow\quad n_1 = n_3,\;\; n_2 = -n_3$$

Thus a normalised eigenvector is: $\qquad \underline{\hat{n}}^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For $t = t^{(3)} = 5$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad n_1 = n_2, \;\; n_3 = 0$$

Thus a normalised eigenvector is: $\qquad \underline{\hat{n}}^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Clearly

$$\begin{aligned} \underline{\hat{n}}^{(1)} \cdot \underline{\hat{n}}^{(2)} &= \frac{1}{\sqrt{18}}(-1 - 1 + 2) \;=\; 0 \\ \underline{\hat{n}}^{(1)} \cdot \underline{\hat{n}}^{(3)} &= \frac{1}{\sqrt{12}}(-1 + 1 + 0) \;=\; 0 \\ \underline{\hat{n}}^{(2)} \cdot \underline{\hat{n}}^{(3)} &= \frac{1}{\sqrt{6}}(1 - 1 + 0) \;=\; 0 \end{aligned}$$

Now suppose that $\lambda_{ij} = n_j^{(i)}$ then

$$\underline{\underline{\lambda}} = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\underline{\underline{\lambda}}\,\underline{\underline{\lambda}}^T = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\lambda}}\,\underline{\underline{T}}\,\underline{\underline{\lambda}}^T = \underline{\underline{\lambda}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 2 \end{pmatrix} \underline{\underline{\lambda}}^T \;=\; \underline{\underline{\lambda}} \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} & \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} & \frac{5}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Thus by choosing an orthonormal basis corresponding to the normalised eigenvectors of the tensor, we find that the tensor is diagonal. This procedure of transforming to the basis provided by the normalised eigenvectors is referred to as *diagonalisation*.

8 *Calculate the gradient* $\underline{\nabla} f$ *for* $f(\underline{r}) = x^2+3y^2+2z^2$. *At the point* $(2,3,1)$ *calculate (a) the gradient to the level surface; (b) the unit normal; (c) the equation of the tangent plane; (d) the directional derivative in the direction of the vector* $2\underline{e}_1 - \underline{e}_2$; (e) the maximum value of f subject to the condition $r^2 = 1$.

For the given scalar field

$$\underline{\nabla} f \;=\; \left(\underline{e}_1 \frac{\partial}{\partial x} + \underline{e}_2 \frac{\partial}{\partial y} + \underline{e}_3 \frac{\partial}{\partial z}\right)(x^2+3y^2+2z^2) \;=\; 2x\underline{e}_1 + 6y\underline{e}_2 + 4z\underline{e}_3$$

(a) From this expression for $\underline{\nabla} f$ we find

$$\underline{\nabla} f \;=\; 4\underline{e}_1 + 18\underline{e}_2 + 4\underline{e}_3 \quad \text{at the point } (2,3,1)$$

(b) To find the unit normal we divide $\underline{\nabla} f$ by its magnitude:

$$\underline{\hat{n}} \;=\; \frac{\underline{\nabla} f}{|\underline{\nabla} f|} \;=\; \frac{1}{\sqrt{89}}(2\underline{e}_1 + 9\underline{e}_2 + 2\underline{e}_3)$$

(c) The equation of the tangent plane is the equation of a plane with normal $\underline{\hat{n}}$ and containing the point $\underline{r}_0$. Thus

$$\underline{r}\cdot\underline{\hat{n}} \;=\; \underline{r}_0\cdot\underline{\hat{n}} \quad\Rightarrow\quad 2x+9y+2z \;=\; 33$$

(d) The directional derivative along $\underline{a}$ is $\underline{\hat{a}}\cdot\underline{\nabla} f$, where $\underline{a} = 2\underline{e}_1 - \underline{e}_2$. Thus

$$\underline{\hat{a}} \;=\; \frac{1}{\sqrt{5}}(2\underline{e}_1 - \underline{e}_2) \quad\Rightarrow\quad \underline{\hat{a}}\cdot\underline{\nabla} f \;=\; -\frac{10}{\sqrt{5}} \;=\; -2\sqrt{5}$$

where we have used the result of part (a) for $\underline{\nabla} f$ at $(2,3,1)$.

(e) If the constraint function is $g(\underline{r}) = r^2 = 1$, then we introduce a Lagrange multiplier λ and look for $\underline{\nabla}(f+\lambda g) = 0$ (note it would have been messier if we had taken $g(\underline{r}) = |\underline{r}| = 1$). $f+\lambda g = (1+\lambda)x^2+(3+\lambda)y^2+(2+\lambda)z^2$, so $\underline{\nabla}(f+\lambda g) = 2[(1+\lambda)x,(3+\lambda)y,(2+\lambda)z]$. This could vanish at the origin (but only for the case $g = 0$); otherwise we need $\lambda = -1, -3$ or -2. These stationary points correspond to $y = z = 0$, $x = z = 0$, $x = y = 0$ respectively, with the remaining coordinate being of unit magnitude in each case (using $g = 1$). Considering each of these in turn, the maximum value of f comes when $y^2 = 1$, so the conditional maxima are at $y = \pm 1$, $f = 3$.

9 (i) *Evaluate the line integral*

$$\int_C \underline{F}\cdot\underline{dr}$$

with $\underline{F} = (y,-x,0)$, *from the point* $(a,0,0)$ *to the point* $(a,0,2\pi b)$ *along a) a circular helix, parameterized by*

$$\underline{r} = (a\cos\lambda,\; a\sin\lambda,\; b\lambda)$$

.

b) a straight line, parameterized by

$$\underline{r} = (a,\ 0,\ b\lambda)\ .$$

(ii) *Repeat for the field $\underline{F} = \underline{r}$. and comment on the dependence of the integral on path.*

(i) a) The end points of the path correspond to $\lambda = 0$ and $\lambda = 2\pi$ respectively. With the given parameterization, we have $x = a\cos\lambda$ and $y = a\sin\lambda$ on the helix so that

$$\begin{aligned} \underline{F} &= (a\sin\lambda,\ -a\cos\lambda,\ 0) \\ \underline{r} &= (a\cos\lambda,\ a\sin\lambda,\ b\lambda) \\ d\underline{r} &= (-a\sin\lambda,\ a\cos\lambda,\ b)d\lambda \end{aligned}$$

Thus

$$\underline{F}\cdot d\underline{r} = (-a^2\sin^2\lambda - a^2\cos^2\lambda)\, d\lambda \ = \ -a^2\, d\lambda$$

and we can write the line integral as

$$\int_C \underline{F}\cdot d\underline{r} \ = \ -a^2\int_0^{2\pi} d\lambda \ = \ -2\pi a^2$$

b) The straight line between the point $(a, 0, 0)$ to the point $(a, 0, 2\pi b)$ is parameterized as

$$\underline{r} \ = \ (a,\ 0,\ b\lambda) \quad \text{where} \quad \lambda : 0 \longrightarrow 2\pi$$

Thus $d\underline{r} \ = \ (0,\ 0,\ b)d\lambda$, $\underline{F} = (y, -x, 0) = (0, -a, 0)$ and hence $\underline{F}\cdot d\underline{r} \ = 0$.

(ii) a) This time $\underline{F} = \underline{r}$ and so

$$\begin{aligned} \underline{F} &= (a\cos\lambda,\ a\sin\lambda,\ b\lambda) \\ d\underline{r} &= (-a\sin\lambda,\ a\cos\lambda,\ b)d\lambda \end{aligned}$$

so that

$$\underline{F}\cdot d\underline{r} \ = (-a^2\cos\lambda\sin\lambda + a^2\cos\lambda\sin\lambda + b^2\lambda)\, d\lambda$$

giving

$$\int_C \underline{F}\cdot d\underline{r} \ = \ b^2\int_0^{2\pi}\lambda d\lambda \ = \ b^2\left[\frac{\lambda^2}{2}\right]_0^{2\pi} \ = \ 2\pi^2 b^2$$

b) For the straight line path with $\underline{F} = \underline{r}$

$$\begin{aligned} \underline{r} &= (a,\ 0,\ b\lambda) \\ d\underline{r} &= (0,\ 0,\ b)d\lambda \\ \underline{F} &= (a,\ 0,\ b\lambda) \end{aligned}$$

so that

$$\int_C \underline{F} \cdot d\underline{r} \;=\; b^2 \int_0^{2\pi} \lambda d\lambda \;=\; 2\pi^2 b^2$$

as before.

We see that the line integral in part (i) is path dependent but in part (ii) is path *independent*. In part (i) the field has non-zero curl:

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\underline{e}_3$$

whereas in part (ii) the field $\underline{F} = \underline{r}$ is *irrotational*:

$$\underline{\nabla} \times \underline{r} = 0$$