FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 9: surface and volume integrals

9.1 If $\underline{A} = 2x^2y\,\underline{e}_1 + xz\,\underline{e}_2 - y^3\,\underline{e}_3$, evaluate the surface integral $\int_S \underline{A} \cdot \underline{dS}$ where S is the surface of the unit cube bounded by: $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Check your result using the divergence theorem:

$$\int_S \underline{A} \cdot \underline{dS} \;=\; \int_V \underline{\nabla} \cdot \underline{A}\, dV.$$

9.2 Evaluate $\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ where S is the *open* hemispherical surface $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, and

$$\underline{F} = (1 - ay)\,\underline{e}_1 + 2y^2\,\underline{e}_2 + (x^2 + 1)\,\underline{e}_3.$$

(Take the vector element of area $\underline{dS}$ to point away from the origin).

Now repeat the evaluation using the divergence theorem applied to the vector field $\underline{\nabla} \times \underline{F}$. (Recall that the divergence theorem applies to a *closed* surface).

9.3 Use the divergence theorem to compute $\int_S \underline{F} \cdot \underline{dS}$ for a vector field $\underline{F} = xy^2\underline{e}_1 + 2y^2\underline{e}_2 + xy^3\underline{e}_3$ over a closed cylindrical surface bounded by $x^2 + z^2 = 4$ and $-1 \leq z \leq 1$.

9.4 If $f(\underline{r})$ is a scalar field defined in a volume V, bounded by a closed surface S, show that

$$\int_V (\underline{\nabla} f)\ dV = \int_S f\ \underline{dS}$$

Hint: apply the divergence theorem to the vector field $\underline{A} = f(\underline{r})\,\underline{c}$, where $\underline{c}$ is an arbitrary *constant* vector.