FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 8: further vector calculus, plus integrals over lines and volumes

8.1 Given a vector field $\underline{A} = xyz(x-y)\underline{e}_1 + xyz(x+y)\underline{e}_2 + xyz^2\underline{e}_3$ compute $\underline{\nabla} \cdot \underline{A}$ and $\underline{\nabla} \times \underline{A}$.

8.2 It is a general result that $\underline{\nabla} \times (\underline{\nabla}\phi) = 0$ for any scalar function $\phi(\underline{r})$. Explain how to prove this result using the symmetry properties of ϵ_{ijk}, when the result is written using components.

By writing the expression for its components, prove that $\underline{\nabla} \times (\phi\,\underline{A}) = \phi\underline{\nabla} \times \underline{A} + (\underline{\nabla}\phi) \times \underline{A}$, where ϕ is a scalar field and $\underline{A}$ is a vector field. Hence show that $\underline{\nabla} \times (\phi\,\underline{\nabla}\psi) = (\underline{\nabla}\phi) \times (\underline{\nabla}\psi)$, where ϕ and ψ are both scalar fields.

8.3 Which of the following vector fields are irrotational? Find a scalar potential for those that are.

(i)	$\underline{A} = \underline{a}$	(iv)	$\underline{A} = \underline{a} \times \underline{r}$
(ii)	$\underline{A} = (\underline{a} \cdot \underline{r})\underline{a}$	(v)	$\underline{A} = \underline{a}f(r)$
(iii)	$\underline{A} = (\underline{a} \cdot \underline{r})\underline{r}$	(vi)	$\underline{A} = \underline{r}f(r)$

In the above: $\underline{a}$ is a constant vector and $f(r)$ is a function of r, not of $\underline{r}$.

8.4 (i) Evaluate the line integral

$$\int_C \underline{F} \cdot \underline{dr}$$

with $\underline{F} = (z, x, -y)$, from the point $(a, 0, 0)$ to the point $(a, 0, 2\pi b)$ along the following curves C:
a) a circular helix, parameterized by $\underline{r} = (a\cos\lambda,\ a\sin\lambda,\ b\lambda)$.
b) Two straight line segments, with $(a, c, 0)$ as an intermediate point between the same starting and finishing points.

(ii) Repeat for $\underline{F} = (z, 0, x)$ and comment on the dependence of the integral on path.

8.5 Consider a hemisphere of radius a symmetric around the $\underline{e}_3$ axis and with its bottom face at $z = 0$. What is its total mass $M = \int_V dV\ \rho(r)$, for $\rho(r) = \sigma/|r|^n$? Compute the centre–of–mass vector $\underline{R} = \int_V dV\ \underline{r}\ \rho(r)/M$.