FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 7: gradient and divergence

7.1 Describe the level surfaces and calculate the gradient $\underline{\nabla} f$ for the following real scalar fields.

$$\begin{aligned}
&\text{(i)} && f(\underline{r}) = x + 2y - 3z \\
&\text{(ii)} && f(\underline{r}) = (x^2 + y^2)^{-1} \\
&\text{(iii)} && f(\underline{r}) = \cos^{-1}(x^2 + y^2 - z)
\end{aligned}$$

Hint: for parts (ii) and (iii) you will have to use the chain rule $\underline{\nabla} F(\phi) = \dfrac{dF}{d\phi}\underline{\nabla}\phi$. For part (iii) try considering $\cos f$.

For the first two fields only, calculate the divergence of $\underline{\nabla} f$.

7.2 Consider the function $f(\underline{r}) = xy + z^2$. (a) At the point $\underline{P} = (3, 2, 1)$ calculate the gradient $\underline{\nabla} f$; (b) give the equation of the plane that is tangent to the level surface at this point; (c) compute the directional derivative at this point in the direction of the vector $2\underline{e}_1 - \underline{e}_3$; (d) Find the stationary points of f subject to the constraint of considering only points that lie in the plane that passes through $\underline{P}$ and is perpendicular to the radius vector at that point (hint: use a Lagrange multiplier).

7.3 The temperature of points in space is given by $T(\underline{r}) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it *cool* as quickly as possible. In which direction should it set out?

Another mosquito is flying at a speed of 5ms^{-1} in the direction of the vector $(4, 4, -2)$. What is its rate of increase of temperature as it passes through the point $(1, 1, 2)$?

7.4 Calculate $\underline{\nabla} f$ for the following forms of the scalar field $f(\underline{r})$, defined in terms of the position vector $\underline{r}$ and its magnitude r. The vector $\underline{a}$ is a constant. Use the chain rule where you can and also the result that $\underline{\nabla}(fg) = (\underline{\nabla} f)g + f\underline{\nabla} g$.

(i)	$f(\underline{r}) = \underline{a} \cdot \underline{r}$	(v)	$f(\underline{r}) = \lvert\underline{r} - \underline{a}\rvert^{-1}$
(ii)	$f(\underline{r}) = (\underline{a} \cdot \underline{r})^n$	(vi)	$f(\underline{r}) = (\underline{a} \times \underline{r})^2$
(iii)	$f(\underline{r}) = \lvert\underline{r} - \underline{a}\rvert^2$	(vii)	$f(\underline{r}) = 3(\underline{a} \cdot \underline{r})^2 - a^2 r^2$
(iv)	$f(\underline{r}) = \lvert\underline{r} - \underline{a}\rvert$	(viii)	$f(\underline{r}) = (\underline{a} \cdot \underline{r})/r^2$

Hint for (vi): use the triple product identities to find f in terms of expressions for which you have already calculated the gradient.

Describe the level surfaces for each of the above fields. (Hint for last three: choose $\underline{a} = a\underline{e}_3$)