FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 6: eigenvectors and diagonalisation of tensors

6.1 Three uniform rods of masses m, m, and $2m$ and lengths $2a$, $2a$ and $4a$ lie along the orthogonal axes Ox_1, Ox_2 and Ox_3 respectively, with one end of each at O. Find the inertia tensor of the system at O, and also at the centre of mass G, referred to axes parallel to Ox_1, Ox_2 and Ox_3. *Verify* that the direction $(1,\,-1,\,0)$ is a principal axis of inertia at G, and find the corresponding moment of inertia.

6.2 A molecule consists of two particles of mass m placed at vector positions (a, a, a) and $(-a, -a, -a)$ relative to an origin O. Show that the inertia tensor is

$$\underline{\underline{I}}(O) = 2\,m\,a^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} .$$

Calculate the eigenvalues and eigenvectors of this inertia tensor, and use symmetry arguments to interpret your results.

6.3 A symmetric second-rank tensor $\underline{\underline{T}}$ has components $T_{ij} = a_i b_j + b_i a_j$ in a particular basis, where a_i and b_i are components of the *unit* vectors $\underline{a}$ and $\underline{b}$.

(i) By calculating $T_{ij}a_j$ and $T_{ij}b_j$, deduce that $\underline{a}$ and $\underline{b}$ are *not* eigenvectors of T. [*i.e.* $T_{ij}\,a_j \neq (\text{constant})\,a_i$, *etc.*]

(ii) Deduce that $(\underline{a} + \underline{b})$ and $(\underline{a} - \underline{b})$ *are* eigenvectors of T, with eigenvalues $(\cos\theta \pm 1)$, where θ is the angle between $\underline{a}$ and $\underline{b}$.

(iii) Verify that $\underline{c} = \underline{a} \times \underline{b}$ is also an eigenvector of T, and find its eigenvalue.

(iv) Check that the sum of the eigenvalues is the trace of T_{ij}.

6.4 (i) Find all eigenvalues and the corresponding set of (orthonormal) eigenvectors of the tensor

$$\underline{\underline{T}} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 2 \end{pmatrix} .$$

Do the eigenvectors form a left–handed or right–handed basis?

(ii) If your eigenvectors form a right–handed (left–handed) basis, how can you transform them into a left–handed (right–handed) basis?

(iii) Use the eigenvectors of a right–handed basis to write down the transformation matrix $\underline{\underline{\lambda}}$ that diagonalizes $\underline{\underline{T}}$.