FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 5: tensors and their transformations

5.1 By considering the transformation matrices that describe two successive rotations through angles θ, ϕ, about, for example, the $\underset{\sim}{e}_3$ axis, establish the identities

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$
$$\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

Show that the same relations are obtained more simply by representing a unit 2D vector by the complex number $\exp(i\theta) = \cos\theta + i\sin\theta$.

5.2 (i) The transformation from one set of basis vectors $\{e_i\}$ into another set $\{e_i'\}$ can be described by matrix coefficients λ_{ij}. If both old and new basis vectors are othonormal, prove that the transformation matrix must be an orthogonal matrix.

(ii) Write down the equation describing how the components of a second-rank tensor change under the above transformation. Show that the quantities T_{ii} and $T_{ij}T_{ji}$ behave as scalars under the transformation. Suggest one further scalar quantity that can be made out of the components of the tensor.

5.3 The moment of inertia of a uniform circular disc of mass M and radius a about a perpendicular axis through its centre is $\frac{1}{2}Ma^2$.

(i) If $\underset{\sim}{e}_3$ is normal to the disc, argue by symmetry that the inertia tensor at the centre O is $\underset{\approx}{I}(O) = \frac{1}{4}Ma^2\text{diag}(1,\,1,\,2)$.

(ii) Use the parallel axis theorem to find $I_{ij}(P)$, where P is a point on the perimeter of the disc [e.g. at $(a,\,0,\,0)$].

(iii) If the disc is spinning with angular velocity $\underline{\omega}$ about an axis through its centre, inclined at an angle θ to the normal [e.g. $\underline{\omega} = \omega(\sin\theta,\,0,\,\cos\theta)$], the angular momentum $\underline{L}(O)$ about O has components $L_i(O) = I_{ij}\,\omega_j$. Find $\underline{L}(O)$ and deduce that it is not, in general, parallel to $\underline{\omega}$.

(iv) By evaluating $\underline{L}(O) \times \underline{\omega}$, find an expression for $\sin\phi$, where ϕ is the angle between $\underline{L}(O)$ and $\underline{\omega}$. Sketch ϕ for θ from $0°$ to $90°$.

5.4 In a certain basis the conductivity tensor of a material has measured components

$$G_{ij} = g\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Find an electric field direction which produces no current and show that in any direction orthogonal to this the current flows equally easily.

[Hint: In effect, this question asks you to find a vector $\underline{E}$ such that $G_{ij}E_j = 0$ and to show that for any vector $\underline{F}$, satisfying $\underline{E}\cdot\underline{F} = 0$, $G_{ij}F_j = \lambda F_i$. You should find $\lambda = 2g$.]