FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 4: more on matrices, index manipulation and change of basis

4.1 Matrix multiplication may be written as follows in index notation: $(\underline{\underline{M}}\ \underline{V})_i = M_{ij}V_j$; $(\underline{\underline{M}}\ \underline{\underline{N}})_{ij} = M_{ik}N_{kj}$.

(a) Verify the latter relation by performing direct matrix multiplication and by writing out the sum in full, for

$$\underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \underline{\underline{N}} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

(b) Use the component definition to show that $[\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}})]_{ij} = A_{ik}B_{k\ell}C_{\ell j}$ and thus prove that matrix multiplication is associative.

(c) The transpose of a matrix has components $(M^T)_{ij} = M_{ji}$. Use this definition to prove that $(\underline{\underline{M}}\ \underline{\underline{N}})^T = \underline{\underline{N}}^T\ \underline{\underline{M}}^T$.

4.2 (i) The transformation from one set of orthonormal basis vectors $\{e_i\}$ into another set $\{e_i'\}$ can be described by matrix coefficients λ_{ij} as follows: $\underline{e}_i' = \lambda_{ij}\underline{e}_j$. Explain what is meant by the term 'orthonormal basis' and show that

$$\lambda_{ij} = \underline{e}_i' \cdot \underline{e}_j \quad \text{and} \quad \lambda_{ik}\lambda_{jk} = \delta_{ij}.$$

Rewrite the last equation using matrix notation.

(ii) If the transformation λ_{ij} is followed by a further transformation μ_{ij}, show that the overall effect can be described by a single matrix.

(iii) Derive the components of λ_{ij} for a rotation by an angle θ about the z axis, taking care to describe the direction of rotation. If this is followed by a second rotation by an angle ϕ, show that the resulting transformation matrix correctly describes a single rotation.

4.3 Construct a right-handed orthonormal basis in which the first vector $\underline{e}_1{}'$ is parallel to $\underline{a} = (1, 1, 1)$ and the second $\underline{e}_2{}'$ is parallel to $\underline{b} = (1, -1, 0)$; and a *left-handed* orthonormal basis in which the first vector $\underline{e}_1{}''$ is parallel to $\underline{c} = (4, -3, 12)$ and the third $\underline{e}_3{}''$ is parallel to $\underline{d} = (-3, 12, 4)$. {Note that $3^2 + 4^2 + 12^2 = 13^2$.}

Calculate the coefficients λ_{ij} of the transformation matrix from the $\underline{e}_i{}'$ basis to the $\underline{e}_i{}''$ basis (using $\lambda_{ij} = \underline{e}_i{}'' \cdot \underline{e}_j{}'$). Verify that its determinant is -1.

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4.4 Let $\underline{A} = a(\underline{e}_1 + \underline{e}_2)$ and $\underline{B} = b(\underline{e}_2 - \underline{e}_1)$ and let the basis $\{\underline{e}_i{}'\}$ be related to the basis $\{\underline{e}_i\}$ basis through

$$\begin{aligned} \underline{e}_1{}' &= \tfrac{1}{3}\,(2\underline{e}_1 + \underline{e}_2 - 2\underline{e}_3) \\ \underline{e}_2{}' &= \tfrac{1}{3}\,(2\underline{e}_1 - 2\underline{e}_2 + \underline{e}_3) \\ \underline{e}_3{}' &= \tfrac{1}{3}\,(\underline{e}_1 + 2\underline{e}_2 + 2\underline{e}_3) \end{aligned}$$

(i) Construct the transformation matrix; is the transformation proper or improper?

(ii) Calculate the transformed components of $\underline{A}, \underline{B}$.

(iii) Calculate the components of $\underline{A} \times \underline{B}$: *firstly* by using the transformed components of $\underline{A}, \underline{B}$ to calculate the vector product; *secondly* by calculating the components of $\underline{A} \times \underline{B}$ in the old basis then transforming. Comment on your results.