FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 3: suffix notation, summation convention and change of basis

3.1 Simplify the following expressions, where δ_{ij} is the Kronecker delta symbol. The summation convention is implied.

(i) $A_i\,\delta_{ij}$; (ii) $A_i\,B_j\,\delta_{ij}$; (iii) $\delta_{ij}\,\delta_{jk}$; (iv) $\delta_{ij}\,\delta_{jk}\,\delta_{k\ell}$; (v) $A_i\,B_j\,C_k\,D_\ell\,\delta_{ij}\,\delta_{k\ell}$

3.2 A set of right-handed orthonormal basis vectors obeys the relation $\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\underline{e}_k$: show that this definition is consistent with the usual xyz coordinate system, and in particular that the basis vectors obey a cyclic pattern $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$ etc.

(a) Hence derive the components of the vector product: $(\underline{A} \times \underline{B})_k = \epsilon_{ijk}A_iB_j$.

(b) Hence re-express the scalar triple product: $\underline{C} \cdot (\underline{A} \times \underline{B}) = \epsilon_{ijk}A_iB_jC_k$.

3.3 Verify that the definition $\det M \equiv M_{1p}M_{2q}M_{3r}\ \epsilon_{pqr}$ correctly gives the determinant of $\underline{\underline{M}}$, an arbitrary 3×3 matrix (remember to assume the summation convention).

3.4 The two sets of basis vectors $\{\underline{e}_i\}$ and $\{\underline{e}_i{}'\}$ are both right-handed orthonormal triads such that $\underline{e}_1{}'$ is in the direction of $(\underline{e}_1 + 2\underline{e}_3)$ and $\underline{e}_2{}'$ is in the direction of $(2\underline{e}_1 + \underline{e}_2 - \underline{e}_3)$.

(i) Construct the correctly normalised basis vectors $\underline{e}_1{}', \underline{e}_2{}', \underline{e}_3{}'$.

(ii) Write down the transformation matrix $\underline{\underline{\lambda}}$ from the basis $\{\underline{e}_i\}$ to the basis $\{\underline{e}_i{}'\}$.

(iii) The vector $\underline{A}$ has components $(1, 2, 3)$ in the original basis; the vector $\underline{B}$ has components $(1, 2, 3)$ in the *new* basis. For each vector, deduce components in both bases, and hence show directly that $\underline{A} \cdot \underline{B}$ is unchanged by the transformation.