FoMP: Vectors, Tensors and Fields 2009/2010

Problem Sheet 1: basic vector manipulation

1.1 Consider the vectors $\underline{a} = (1, 2, 4)$ and $\underline{b} = (1, 1, 3)$ in an orthonormal, right-handed basis.

(i) construct *unit* vectors $\hat{\underline{a}}$ parallel to $\underline{a}$ and $\hat{\underline{b}}$ parallel to $\underline{b}$;

(ii) use the scalar product to determine the component of $\underline{a}$ in the direction of $\underline{b}$;

(iii) hence express $\underline{a}$ as a sum of vectors parallel and perpendicular to $\underline{b}$;

(iv) construct a vector $\underline{c}$ orthogonal to both $\underline{a}$ and $\underline{b}$ by requiring that $\underline{c}\cdot\underline{a} = \underline{c}\cdot\underline{b} = 0$;

(v) Verify that $\hat{\underline{c}}$ is proportional to $\pm\hat{\underline{a}} \times \hat{\underline{b}}$. What is the coefficient of proportionality, and what determines the sign?

1.2 Find the general vector that is orthogonal to $\underline{a}$ and coplanar with $\underline{a}$ and $\underline{b}$ where $\underline{a} = (8, 2, -1)$ and $\underline{b} = (3, -1, -3)$ in an orthonormal basis.

1.3 A (plane) triangle is formed by the three vectors $\underline{A}$, $\underline{B}$ and $\underline{C}$, such that

$$\underline{A} + \underline{B} + \underline{C} = 0 \,,$$

and the angles at the vertices are α (between $\underline{B}$ and $\underline{C}$), β (between $\underline{C}$ and $\underline{A}$) and γ. Show that $\underline{A} \times \underline{B} = \underline{B} \times \underline{C} = \underline{C} \times \underline{A}$, and hence establish the 'sine rule' that:–

$$\frac{\sin\alpha}{|\underline{A}|} = \frac{\sin\beta}{|\underline{B}|} = \frac{\sin\gamma}{|\underline{C}|} \,.$$

1.4 Which of the following vector identities are true?

(i) $\underline{c} \cdot (\underline{a} \times \underline{b}) = (\underline{b} \times \underline{a}) \cdot \underline{c}$;

(ii) $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c}$;

(iii) $(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$;

(iv) $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{b}) = \underline{b}[\underline{b} \cdot (\underline{c} \times \underline{a})]$.

1.5 Deduce the folowing vector identities:

(i) $\underline{A} \times (\underline{B} \times \underline{C}) + \underline{B} \times (\underline{C} \times \underline{A}) + \underline{C} \times (\underline{A} \times \underline{B}) \;=\; 0$;

(ii) $(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) \;=\; (\underline{A} \cdot \underline{C})(\underline{B} \cdot \underline{D}) - (\underline{A} \cdot \underline{D})(\underline{B} \cdot \underline{C})$;

(iii) $(\underline{A} \times \underline{B}) \times (\underline{C} \times \underline{A}) \;=\; -(\underline{A}, \underline{B}, \underline{C})\underline{A}$.

Note: question 4 & 5 involve material from lecture 3.