STATISTICAL PHYSICS 06/07

The Ising Model **Tutorial Sheet 6**

The questions that follow on this and succeeding sheets are an integral part of this course. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

6.1 **Mean-Field Theory of a Three State Model [S]**

A set of spins on a lattice of coordination number z can take values $S_i = (1, 0, -1)$, as opposed to just $(1, -1)$ as in the Ising model. The configurational energy is (as usual)

$$E(\{S_i\}) = -J \sum_{<ij>} S_i S_j - h \sum_i S_i$$

Show that in the mean-field approximation, the magnetization per site obeys

$$m = \frac{2\sinh(\beta(Jzm + h))}{2\cosh(\beta(Jzm + h)) + 1}$$

and find the critical temperature, T_c.

6.2 **Critical Exponents from Mean-Field Theory of Ising [R/S]**

(i) If we denote the order parameter (or magnetisation per spin) by m, show that the mean-field solution of the Ising model can be written as

$$m = \tanh\left[\frac{m}{1+t} + b\right],$$

where $t = (T - T_c)/T_c$ is the reduced temperature and $b = h/kT$ is the reducedexternal magnetic field. By considering m for temperature close to T_c and for zero external field, show that the associated critical exponent takes the value $\beta = \frac{1}{2}$.

[Hint: the following expansion

$$\tanh x = x - \frac{1}{3}x^3 + O(x^5),$$

for small values of x, should be helpful.]

(ii) Obtain an expression for the mean energy $\overline{E}$ of the Ising model when the applied field is zero, using the simplest mean-field approximation. Hence show that the heat capacity C_h has the behaviour:

$$\begin{aligned} C_h &= 0, && \text{for} \quad T > T_c\,; \\ &= 3Nk/2, && \text{for} \quad T < T_c\,. \end{aligned}$$

Comment on the nature of the singularity at T_c. Is there a critical exponent α?

(iii) By considering the behaviour of the order parameter at temperatures just above the critical point, show that the critical exponent γ, which is associated with the isothermal susceptibility, takes the value $\gamma = 1$ according to mean-field theory.

Also, by considering the effect of an externally applied mangetic field at $T = T_c$, show that the exponent associated with the critical isotherm takes the value $\delta = 3$.

6.3 **The Lattice Gas [S]** Check the claim, made in the notes, that the Energy of the lattice gas model in the grand canonical ensemble

$$E'(\{c\}) = \epsilon \sum_{<ij>} c_i c_j - \mu \sum_i c_i$$

is transformed to that of the Ising model:

$$E'(\{s\}) = -J \sum_{<ij>} S_i S_j - h \sum_i S_i + \frac{\epsilon z - 4\mu}{8} \mathcal{N}$$

where $\mathcal{N}$ is the number of sites, by means of the change of variable

$$S_i = 2c_i - 1 = \pm 1 \quad ; \quad -J = \frac{\epsilon}{4} \quad ; \quad -h = \frac{\epsilon z - 2\mu}{4}$$

6.4 **Transfer Matrix Solution of the one-dimensional Ising Model [C]**

Show that the canonical partition function of the one-dimensional Ising model (with periodic boundary conditions) may be written as

$$Z_c = \sum_{S_1=\pm1} \sum_{S_2=\pm1} \cdots \sum_{S_N=\pm1} T(S_1, S_2) T(S_2, S_3) \ldots T(S_N, S_1)$$

where

$$T(S_i, S_{i+1}) = \exp\left[\beta J S_i S_{i+1} + \frac{\beta h}{2}(S_i + S_{i+1})\right]$$

Hence show that

$$Z_c = \lambda_+^N + \lambda_-^N \simeq \lambda_+^N$$

where $\lambda_\pm$ are the eigenvalues, which you should determine, of the transfer matrix $\hat{T}$

$$\hat{T} = \begin{pmatrix} T(1,1) & T(1,-1) \\ T(-1,1) & T(-1,-1) \end{pmatrix}$$

Show that the magnetisation is zero for $h = 0$ at all temperatures T and find an expression for the mean energy at low temperature.