# STATISTICAL PHYSICS

**Phonons; Virial Expansion** **Tutorial Sheet 4**

The questions that follow on this and succeeding sheets are an integral part of this course. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

4.1 **Debye Model with a different density of states [S]**

Consider a 3d solid consisting of $N$ atoms where the density of modes is

$$g(\omega) = b\omega^4$$

where $b$ is a constant. The frequencies range from zero to some cut-off $\omega_{\rm max}$.

Find an expression for $\omega_{\rm max}$.

Use $\omega_{\rm max}$ to define a characteristic temperature and identify high $T$ and low $T$ regimes

Calculate the total energy $\overline{E}$ and the heat capacity in the low and high temperature limits, which you should define. Express your results purely in terms of $N, h, k, T$ and $b$ (and a dimensionless integral where required).

4.2 **High and low temperature limits and corrections to Debye model [s/c]**

First think about the density of states for a particle in a box in $d$ dimensions and show that it should behave as $g(\omega) \propto \omega^{d-1}$.

**a)** Using this form of the density of states for the Debye model of phonons in a $d$-dimensional isotropic solid containing N atoms, show that

$$\overline{E} = \frac{N d^2 \hbar\omega_D}{\alpha^{(1+d)}} f(\alpha)$$

where $\alpha = \dfrac{\hbar\omega_D}{kT}$ and

$$f(\alpha) = \int_0^\alpha {\rm d}x \frac{x^d}{{\rm e}^x - 1} \, .$$

**b)** In the high temperature limit $kT \gg \hbar\omega_D$ calculate the leading behaviour of $C_V$ and first correction to this behaviour.

**c)** Show that

$$f(\infty) = \zeta(d+1)\Gamma(d+1)$$

where the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty u^{z-1} \mathrm{e}^{-u} \mathrm{d}u$$

.

**d)** By expressing $f(\alpha) = f(\infty) - \Delta f(\alpha)$, calculate the leading behaviour of $C_V$ and the first correction to this.

### 4.3 Second Virial Coefficient [S]

Show that for a spherically symmetric potential $\phi(r)$, the expression for the second virial coefficient may be written as

$$B_2 = 2\pi \int r^2 \left[1 - e^{-\phi(r)/kT}\right] dr \,.$$

If a gas of interacting particles is modelled as hard spheres of radius $a/2$ (minimum separation between centres $a$), show that the second virial coefficient takes the form:

$$B_2 = \frac{2\pi a^3}{3} \,.$$

### 4.4 Simple form of Second Virial Coefficient [S]

Assuming that $\phi(r)$ is large (on the scale of $kT$) and positive for $r < r_0$ and small for $r > r_0$, show that the second virial coefficient may be written as

$$B_2(T) = b_0 - \frac{a_0}{kT}$$

where you should obtain expressions for $a_0$ and $b_0$.

Compare this form of $B_2$ with that implied by the Van der Waals equation of state.

Calculate the entropy and show that

$$S = S_{\text{Ideal}} - Nkb_0\rho$$

Comment on why the entropy is reduced from the standpoint of information theory.

### 4.5 Failure of Perturbation Theory for Coulomb Interaction [S]

Consider a system of particles whose interaction potential falls off like $r^{-y}$ as $r \to \infty$. Show that $B_2$ is infinite if $y \leq 3$. What is the physical implication of this?