Statistical Physics: Tutorial 8

John Peacock
Institute for Astronomy, Royal Observatory Edinburgh

Ising Model; Landau Theory; Universality

A: Core problems

1. **Critical exponents from the Ising mean-field theory**

   (a) If we denote the order parameter (or magnetisation per spin) by m, show that the mean-field solution of the Ising model can be written as

   $$m = \tanh\left[\frac{m}{1+t} + b\right],$$

   where $t = (T - T_c)/T_c$ is the reduced temperature and $b = h/kT$ is the reduced external magnetic field. By considering m for temperature close to T_c and for zero external field, show that the associated critical exponent (power-law dependence of m on t) takes the value $\beta = 1/2$ [you may assume that $\tanh x \simeq x - x^3/3 + O(x^5)$].

   (b) Obtain an expression for the mean energy $\overline{E}$ of the Ising model when the applied field is zero, using the simplest mean-field approximation. Hence show that the heat capacity C_h has the behaviour:

   $$\begin{aligned} C_h &= 0, \quad &\text{for} \quad T > T_c\,; \\ &= 3Nk/2, \quad &\text{for} \quad T < T_c\,. \end{aligned}$$

   Comment on the nature of the singularity at T_c. Is there a critical exponent α for C_h?

   (c) By considering the behaviour of the order parameter at temperatures just above T_c, show that the critical exponent γ, associated with the temperature dependence of the isothermal magnetic susceptibility, takes the value $\gamma = 1$ according to mean-field theory.

   (d) Consider the effect of an externally applied magnetic field at $T = T_c$, and show that m scales as the 1/3 power of the field at this temperature.

2. **Free energy of the Ising model**

   (a) Show that the mean-field Ising model has a free energy function per spin

   $$f(m) = -\frac{zJ}{2}m^2 - hm + kT\left[\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right)\right].$$

   (b) Show that minimising $f(m)$ leads to the Weiss mean-field equation for the magnetisation.

3. **Ising model representation of binary alloy**

   The simplest model of a binary allow envisages a system consisting of $N/2$ atoms of species A and $N/2$ atoms of species B, distributed over sites forming a body centred cubic lattice. The energy of interaction of neighbouring pairs of atoms is $\epsilon_{\rm AA}$, $\epsilon_{\rm AB}$ or $\epsilon_{\rm BB}$ according to the pair involved. Show that the canonical partition function is that of the Ising model in zero field and with interaction parameter $J = -(1/4)\,(\epsilon_{\rm AA} + \epsilon_{\rm BB} - 2\epsilon_{\rm AB})$.

4. **Order parameter of an antiferromagnet**

   The simplest representation of an antiferromagnet is afforded by the Ising model with a nearest neighbour interaction parameter J that is *negative*.

(a) By considering a one-dimensional system identify the two possible ground states and hence the two phases that one might expect to coexist at low enough temperatures. Find a suitable order parameter that differentiates between the two low temperature phases and is zero in the high temperature disordered phase.

(b) Generalise to a three-dimensional system.

B: Further problems

1. **The lattice gas** Check the claim in the notes that the energy of the lattice gas model in the grand canonical ensemble,

$$E'(\{c\}) = \epsilon \sum_{<ij>} c_i c_j - \mu \sum_i c_i \ ,$$

is transformed to that of the Ising model:

$$E'(\{s\}) = -J \sum_{<ij>} S_i S_j - h \sum_i S_i + \frac{\epsilon z - 4\mu}{8} \mathcal{N} \ ,$$

where $\mathcal{N}$ is the number of sites, by means of the change of variable

$$S_i = 2c_i - 1 = \pm 1 \quad ; \quad -J = \frac{\epsilon}{4} \quad ; \quad -h = \frac{\epsilon z - 2\mu}{4} \ .$$

2. **Transfer matrix solution of the 1D Ising model**

Show that the canonical partition function of the one-dimensional Ising model (with periodic boundary conditions) may be written as

$$Z_c = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \ldots \sum_{S_N=\pm 1} T(S_1, S_2) T(S_2, S_3) \ldots T(S_N, S_1) \ ,$$

where

$$T(S_i, S_{i+1}) = \exp \left[\beta J S_i S_{i+1} + \frac{\beta h}{2} (S_i + S_{i+1}) \right] \ .$$

Hence show that

$$Z_c = \lambda_+^N + \lambda_-^N \simeq \lambda_+^N \ ,$$

where $\lambda_\pm$ are the eigenvalues, which you should determine, of the transfer matrix $\hat{T}$:

$$\hat{T} = \begin{pmatrix} T(1,1) & T(1,-1) \\ T(-1,1) & T(-1,-1) \ . \end{pmatrix}$$

Show that the magnetisation is zero for $h = 0$ at all temperatures T and find an expression for the mean energy at low temperature.

3. **Critical exponents in Landau theory**

Check and complete the Landau theory calculations, given in lectures, for the critical exponents β, γ, δ and α of the Ising model. For the last of these you will need to prove the result $c_h = -T \left(\partial^2 f / \partial T^2 \right)_h$.