School of Physics & Astronomy

Statistical Physics

PHYS11024 (SCQF Level 11)

Tuesday 0th May, 2024 13:00 - 15:00
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **BOTH** questions

Special Instructions

- Electronic Calculators must **not** be used during this examination.
- Attach supplied anonymous barcodes to *each* script book used.

Special Items

- School supplied barcodes

Chairman of Examiners: Prof. J Dunlop
External Examiner: Prof. D Litim

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1.

(a) Explain what it means to say that a system is part of the Canonical Ensemble, and write down the partition function for this case, Z. Use the Gibbs definition of entropy to derive the relation between Z and the free energy, F. Now consider dZ/dT, and show that it is related to the mean energy of the system, $\bar{E}$. Similarly, show that fluctuations in the energy have a variance of $(kT)^2(C_V/k)$, where C_V is the specific heat at fixed volume. **[5]**

(b) Explain what is meant by the chemical potential, μ, and say how it appears in the Gibbs factor that governs microstate probabilities in systems with variable particle number. Photons can be treated as a gas of non-interacting identical bosons, where the occupation numbers for each wave mode are subject to independent Gibbs factors. Determine the mean occupation number for $\mu \neq 0$, and hence show that the total photon number and total energy in thermal equilibrium are

$$N = \frac{4\pi gV}{c^3}\int_0^\infty \frac{\nu^2\, d\nu}{\exp[(h\nu-\mu)/kT]-1}; \quad E = \frac{4\pi hgV}{c^3}\int_0^\infty \frac{\nu^3\, d\nu}{\exp[(h\nu-\mu)/kT]-1}.$$

For the case of black-body radiation with $\mu = 0$, show that $E \propto T^3$, and hence prove that the entropy is $S = (4/3)E/T$. **[6]**

(c) A set of black-body photons is given a pulse of injected energy, which raises the temperature by a factor $1+\alpha$, but without changing the total number of photons. Assuming that $\alpha \ll 1$, show that the resulting system has a non-zero chemical potential, $\mu \propto -\alpha T$ to lowest order in α. Express the constant of proportionality in terms of $I_n \equiv \int_0^\infty x^n(\exp[x]-1)^{-1}dx$. **[6]**

(d) Consider a model where the Landau free energy takes the form

$$F = am^2/2 + bm^4/4 + cm^6/6,$$

where m is some order parameter and $c > 0$. Sketch the form of this potential in the different quadrants of the (a, b) plane. Hence argue that there is a line of second-order transitions at $(a = 0, b > 0)$, and a line of first-order transitions in the quadrant $(a > 0, b < 0)$. Show that the equation for the latter line is

$$b = -4(ca/3)^{1/2}.$$

If a varies linearly with T and b is independent of T, compute the critical exponent β ($m \propto |t|^\beta$, where t is the fractional deviation in temperature from the critical temperature). **[8]**

2.

(a) Explain what is meant by a Joule expansion of a gas, and say why this occurs at constant temperature in the case of a perfect gas. Hence calculate the entropy change associated with the process. According to the Poincaré Recurrence Theorem, the molecules in a given box will all spontaneously move to one side of the box in a finite time: what does this imply for the correctness of the Second Law of Thermodynamics? **[5]**

(b) In the Ising model, the energy of a set of N spins, $S_i = \pm 1$, in a magnetic field h can be written as

$$E = -J \sum_{\mathrm{z}} S_i S_j - h \sum_i S_i,$$

where the spin interaction is between each spin and its z nearest neighbours. By assuming that fluctuations of spin away from the mean value, m, are weakly correlated, prove that $\langle S_i S_j \rangle \simeq m(S_i + S_j) - m^2$, and hence that

$$E = \frac{1}{2} JNzm^2 - h_{\mathrm{eff}} \sum_i S_i; \quad h_{\mathrm{eff}} = Jzm + h.$$

Show that the partition function is

$$Z = \exp[-\beta JNzm^2/2] \left(\exp[-\beta h_{\mathrm{eff}}] + \exp[\beta h_{\mathrm{eff}}]\right)^N,$$

and hence give the equation that determines m in this mean-field model. **[6]**

(c) For imperfect gases, the partition function can be written as $Z = Z_{\mathrm{ideal}} \times Q$, where $Q = 1$ for non-interacting particles. Show that Q can be expressed as follows in terms of the interaction potential, U, which depends on the positions of all N particles:

$$Q = 1 + V^{-N} \int \prod_i d^3 q_i \left(\exp[-\beta U] - 1\right).$$

If the interactions are strictly pairwise and N is large, explain why this can be approximated as

$$Q = (1 + 2B_2/V)^{N^2/2},$$

where the quantity B_2 is negative if the intermolecular forces are attractive. Hence show that the pressure of a non-ideal gas is approximately $P = nkT(1 + B_2 n)$, where n is the number density. **[6]**

(d) Starting from the chain rule in the form $dz = (\partial z/\partial x)_y \, dx + (\partial z/\partial y)_x \, dy$ and $dz = 0$, prove that

$$(\partial z/\partial x)_y = -(\partial z/\partial y)_x (\partial y/\partial x)_z.$$

Apply this result to derive the temperature change in Joule expansion,

$$(\partial T/\partial V)_E = -\frac{1}{C_V} \left[n^2 k T^2 \frac{\partial B_2}{\partial T} \right].$$

You will need to derive the Maxwell relation $(\partial S/\partial V)_T = (\partial P/\partial T)_V$. **[8]**

Invigilators Information

Delivered papers 30 minutes prior to start time.

1. Course information

Course title:	Statistical Physics		
Course No:	PHYS11024		
Academic :	Prof John Peacock	**Contact No:**	07946 273 597, 01316 674 127
Secretary:	Stavriana Manti	**Contact No:**	0131505949
School:	School of Physics & Astronomy	**Contact No:**	68-8261 / 51-7525

2. Exam diet / paper information

Date of exam:	Tuesday 0^{th} May, 2024	**Time of exam:**	13:00 - 15:00
Location of exam:	Chrystal MacMillan Seminar Room 1/2		
No of exam papers:	33		

3. Invigilators Instructions

Unissued paper returned:	Yes	**Approved Calculators :**	No
Answer on exam paper:	No	**Open Book Exam:**	No
Answer on MCQ :	No	**Script book per answer:**	See rubric
Used exam papers returned:	No	**Dictionary allowed:**	No

4. Stationery Requirement

Stationery: 1 x 16 sides script book.

5. Items to be handed out with exam papers

Calculators (from School):	No	**Barcodes (from School):**	Yes
Graph paper (from School:	No	**Bibles (from School)**:	No
Formula Sheets (from School):	No		

6. Additional Information

- None

Notes: