School of Physics & Astronomy

THE UNIVERSITY of EDINBURGH

Statistical Physics

PHYS11024 (SCQF Level 11)

Friday 5th May, 2023 13:00 - 15:00
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **BOTH** questions

Special Instructions

- Electronic Calculators must **not** be used during this examination.
- Attach supplied anonymous barcodes to *each* script book used.

Special Items

- School supplied barcodes

Chairman of Examiners: Prof. J Dunlop
External Examiner: Prof. D Litim

Anonymity of the candidate will be maintained during the marking of this examination.

1.

(a) If a system is in a microstate i, it will evolve to a definite state j at some later time. Use this fact to explain why the Gibbs entropy is a constant of the motion.

Consider a cell in phase space, and suppose that a fraction f of the volume in the cell has a phase-space density ρ but that the remaining fraction has zero density. Calculate the exact entropy and say how it compares with the entropy that results from using the coarse-grained density averaged over the cell. By using the concept of a convex function, demonstrate that coarse-graining always results in an increase in entropy if the density is not constant within the cell. **[6]**

(b) A given system is described by two variables, x & y, each of which can take a set of discrete values. The Gibbs entropy is

$$S = -k \sum_i \sum_j p(x_i, y_j) \ln p(x_i, y_j).$$

Compare this to the entropy for x alone, using $p(x_i) = \sum_j p(x_i, y_j)$. Show that the difference in entropy (the conditional entropy) can be expressed using the total probability $p(x_i, y_j)$ and the conditional probability of y given x: $p(y|x) = p(x, y)/p(x)$. **[4]**

(c) A quantum system consists of two states: state $+$, with energy $+\epsilon$, and state $-$, with energy $-\epsilon$. Write down the master equation(s) governing the occupation probabilities, p_+ and p_-, in terms of the transition rates $\nu_\uparrow$ and $\nu_\downarrow$ (respectively the rate of transitions to $+$ if the state is $-$, and vice versa). Explain what is meant by the principle of detailed balance and hence derive the relation between $\nu_\uparrow$ and $\nu_\downarrow$ required for equilibrium at temperature T.

Derive the master equation for $\Delta \equiv p_+ - p_-$ in the limit $\epsilon \ll kT \equiv 1/\beta$, and show that it has the solution

$$\Delta(t) = A \exp[-2(1 - \beta\epsilon)\nu_\downarrow t] - \frac{\beta\epsilon}{1 - \beta\epsilon},$$

where A is an undetermined constant. **[6]**

(d) In the Debye model for the specific heat of a uniform solid, there are three wave modes for a given wavevector: two transverse and one longitudinal. Usually these are assumed to have the same speed of propagation, but in reality longitudinal waves are substantially faster. Write the ratio of longitudinal and transverse speeds as α, and explain why both sets of modes should have the same maximum wavenumber, k_m. Hence give an expression for the exact mean energy in the general Debye model as a function of temperature. In the limit of high T, show that the specific heat is

$$C_V \simeq 3Nk \left[1 - \frac{2 + \alpha^2}{60} \left(\frac{\hbar k_m c_s}{kT}\right)^2\right],$$

where c_s is the sound speed for transverse waves. You may assume the following Taylor expansion: $x^3/(\exp[x] - 1) \simeq x^2 - x^3/2 + x^4/12$. **[9]**

2.

(a) For a system that can exchange heat and particles with a heat bath at temperature T, the probability of occupying microstate i is given by a Gibbs factor:

$$p_i \propto \exp[-\beta(E_i - \mu N_i)].$$

Explain the meaning of the terms in this equation, and show that it corresponds to maximisation of entropy subject to two constraints. **[4]**

(b) Define the grand potential, Φ, and explain how the entropy and pressure of a system can be computed if Φ is known. Define the grand partition function, Z_{G}, and derive the bridge equation that relates Z_{G} to Φ. **[4]**

(c) Consider hypothetical identical spinless particles called 'Doubleons', for which quantum statistics permit a maximum of two particles to occupy the same quantum state. A system containing such particles comes into thermal equilibrium with an external system, with which it can exchange energy and particles. Calculate the mean number of particles in a given energy level and compare it with the result for Fermions and Bosons. Explain why the Doubleon result is different from the case of Fermions with spin 1/2, where also two particles can occupy the same energy level. **[4]**

(d) For an ideal quantum gas with volume V, the density of states in momentum space is $gV/(2\pi\hbar)^3$, where g is a spin degeneracy factor. Write down expressions for the mean number of particles, $\bar{N}$, and the pressure, P in thermal equilibrium. In the limit where the occupation numbers are all small, show that this implies $PV = \bar{N}kT$, independent of Fermi/Bose statistics, and of whether or not the particles are non-relativistic. **[5]**

(e) A set of spins on a lattice of coordination number z can take values $S_i = (1, 0, -1)$, as opposed to just $(1, -1)$ as in the Ising model. The configurational energy can be written as

$$E(\{S_i\}) = -J \sum_{<ij>} S_i S_j - h \sum_i S_i \,.$$

Explain the meaning of all the terms in this equation. Show that in the mean-field approximation, the magnetization per site obeys

$$m = \frac{2\sinh(\beta(JNm + h))}{2\cosh(\beta(JNm + h)) + 1},$$

explaining the meaning of the constant N. Hence find the critical temperature, T_c. **[8]**

Invigilators Information

Delivered papers 30 minutes prior to start time.

1. Course information

Course title:	Statistical Physics		
Course No:	PHYS11024		
Academic :	Prof John Peacock	**Contact No:**	07946 273 597, 01316 674 127
Secretary:	Stavriana Manti	**Contact No:**	0131505949
School:	School of Physics & Astronomy	**Contact No:**	68-8261 / 51-7525

2. Exam diet / paper information

Date of exam:	Friday 5th May, 2023	**Time of exam:**	13:00 - 15:00
Location of exam:	Chrystal MacMillan Seminar Room 1/2		
No of exam papers:	33		

3. Invigilators Instructions

Unissued paper returned:	Yes	**Approved Calculators :**	No
Answer on exam paper:	No	**Open Book Exam:**	No
Answer on MCQ :	No	**Script book per answer:**	See rubric
Used exam papers returned:	No	**Dictionary allowed:**	No

4. Stationery Requirement

Stationery: 1 x 16 sides script book.

5. Items to be handed out with exam papers

Calculators (from School):	No	**Barcodes (from School):**	Yes
Graph paper (from School:	No	**Bibles (from School)**:	No
Formula Sheets (from School):	No		

6. Additional Information

- None

Notes: