School of Physics & Astronomy

THE UNIVERSITY of EDINBURGH

Statistical Physics

PHYS11024 (SCQF Level 11)

Monday 29$^{\text{th}}$ April, 2019 14:30 - 16:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **TWO** questions

Special Instructions

- Electronic Calculators must **not** be used during this examination.
- Attach supplied anonymous bar codes to *each* script book used.

Special Items

- School supplied barcodes

Chairman of Examiners: Prof A Trew
External Examiner: Prof E Copeland

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1. **a)** Explain why the average energy $\overline{E}$ of a solid in the harmonic approximation is given by

$$\overline{E} = \int \mathrm{d}\omega\, \hbar\omega\, g(\omega)\, \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}\,,$$

where $\beta = 1/(k_B T)$ and $g(\omega)$ is the density of modes of angular frequency ω. You should explain the meaning of each factor in the integrand. **[5]**

b) In the harmonic approximation the density of phonon modes with wave number k in a three-dimensional solid with $3N$ modes is given by $\Gamma(k) = AVk^2$.

Assuming this relation, show that $g(\omega)$ is given by

$$g(\omega) = \frac{AV}{c^3}\omega^2$$

where $\omega = ck$. **[2]**

c) Show that in Debye theory the average energy becomes

$$\overline{E} = \frac{AV\hbar}{c^3(\beta\hbar)^4}\int_0^{\theta_D/T} \mathrm{d}x\, \frac{x^3}{\exp(x) - 1}\,,$$

where $\theta_D \equiv \hbar\omega_{\mathrm{max}}/k_B$ and ω_{max} is a maximum frequency for which you should derive an expression. **[6]**

d) (i) Determine the leading order expression for $\overline{E}$ and C_V, the heat capacity at constant volume, for the Debye solid in the high-temperature limit $T \to \infty$. **[4]**

(ii) Determine the leading order expression for $\overline{E}$ and C_V for the Debye solid in the low-temperature limit $T \to 0$. (You do not need to evaluate any dimensionless integral that appears.) **[3]**

e) Now consider a different sort of collective mode excitation for which the relation $\omega = ak^2$ holds.

Assuming the harmonic approximation for $\Gamma(k)$ given in part **b)**, deduce the temperature dependence of the contribution to C_V of these excitations at low temperature.

[5]

2. Consider a generalised Ising model with three states $S_i = 1, 0, -1$ for each of N sites labelled by $i = 1, \ldots, N$.

a) The canonical partition function is

$$Z = \sum_{\{S_i\}} \exp[-\beta E(\{S_i\})] \, ,$$

where

$$E(\{S_i\}) = -J \sum_{<ij>} S_i S_j - h \sum_{i=1}^{N} S_i \, .$$

Explain the notation and physical meaning of each of the terms in the above two equations. **[4]**

b) Explain what is meant by a *mean-field* theory and explain why the mean field is $h_{mf} \equiv h + Jzm$ where $m = \langle S_i \rangle$ is the magnetisation per spin and z is the co-ordination number of the lattice. **[3]**

c) Show that in mean-field theory the magnetisation per spin is given by

$$m = \frac{2\sinh(\beta h_{mf})}{2\cosh(\beta h_{mf}) + 1} \, .$$

In your answer you should explain carefully why the mean-field assumption simplifies the problem. **[7]**

d) Show graphically how non-zero solutions for m emerge and deduce that for zero external magnetic field, $h = 0$, the critical temperature T_c for this model is given by

$$T_c = \frac{2Jz}{3k_B} \, ,$$ **[4]**

where k_B is Boltzmann's constant.

e) Compute the leading t dependence of the zero-field susceptibility $\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0}$, where $t = \frac{T - T_c}{T_c}$ is the reduced temperature, in the case of *positive* t, identifying the corresponding critical exponent. **[5]**

f) Now consider a generalised Ising model with four states $S_i = 1, 1/2, -1/2, -1$ and partition function as in part **a)**.

Do you expect T_c to increase or decrease compared to the three-state value? A detailed calculation is not required, but you should give a reasoned explanation of your answer. **[2]**

3. **a)** Discuss the issue of reconciling the arrow of time with physics at the microscopic level. You should clearly state the problem and make reference to the use of the Gibbs entropy and coarse-graining procedures. **[6]**

b) The Langevin equation for a Brownian particle is

$$m\dot{v} = -\gamma v + f(t) \ ,$$

where v is the speed, γ is a constant and f is a random force.

(i) Explain the meaning of the terms in this equation and why $f(t)$ can be taken to obey

$$\begin{aligned} \langle f(t) \rangle &= 0 \\ \langle f(t) f(t') \rangle &= \Gamma \delta(t - t') \ , \end{aligned}$$

where Γ is a constant . **[4]**

(ii) Why does the Langevin equation provide a coarse-grained description of the microscopic physics? **[1]**

c) Now take $m = 1$ and the initial speed as v_0.

(i) Show that

$$\langle v(t) \rangle = v_0 \mathrm{e}^{-\gamma t}$$

and

$$\langle v(t)^2 \rangle = v_0^2 \mathrm{e}^{-2\gamma t} + \frac{\Gamma}{2\gamma} \left[1 - \mathrm{e}^{-2\gamma t} \right] \ .$$ **[7]**

(ii) Give an annotated sketch of $\langle v^2 \rangle$, deducing the short and long-time limits. **[4]**

(iii) Deduce a relation between Γ and γ, stating any principle that you invoke. **[3]**