School of Physics & Astronomy

Statistical Physics

PHYS11024 (SCQF Level 11)

Wednesday 10th May, 2017 14:30 – 16:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **TWO** questions

Special Instructions

- Electronic Calculators must **not** be used during this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied barcodes

Chairman of Examiners: Prof S Playfer
External Examiner: Prof G Shore

Anonymity of the candidate will be maintained during the marking of this examination.

1. The grand potential, Φ, is defined by

$$\Phi = \overline{E} - TS - \mu\overline{N}\,.$$

State the names of the terms in this equation. **[2]**

Show that

$$\begin{aligned}\overline{N} &= -\left.\frac{\partial\Phi}{\partial\mu}\right|_{V,T}\,, \\ \overline{E} &= \left.\frac{\partial}{\partial\beta}(\beta\Phi)\right|_{V,\mu} + \mu\overline{N}\,,\end{aligned}$$

where V is the volume of the system, $\beta = 1/(kT)$ and k is Boltzmann's constant. **[3]**

A system consisting of non-interacting spinless bosons has particle levels labelled by j with energies $\epsilon_j \geq 0$. Show that the grand potential is given by

$$\Phi = kT\sum_j \ln\left(1 - e^{-\beta(\epsilon_j - \mu)}\right)\,.$$

[4]

What is the maximum possible value of μ? (A reason must be given.) **[1]**

A system of spinless non-interacting bosons in a volume V has one level of energy zero and a large number $M \gg 1$ of levels all with energy ϵ. M may be written as $M = AV$ where A is a positive constant.

(a) Show that the grand potential becomes

$$\Phi = kTV\left(\frac{1}{V}\ln\left(1 - e^{\beta\mu}\right) + A\ln\left(1 - e^{-\beta(\epsilon-\mu)}\right)\right)\,.$$ **[2]**

(b) Given a fixed $\rho = \overline{N}/V$ (the particle density) calculate the temperature T_c at which Bose condensation occurs. **[3]**

(c) For $T > T_c$ show that $\mu = \epsilon(T_c - T)/T_c$. What is the value of μ for $T < T_c$? **[4]**

(d) Calculate the average energy $\overline{E}$ for both $T > T_c$ and $T < T_c$. **[3]**

(e) Hence find the specific heat at constant volume C_V also for $T > T_c$ and $T < T_c$. **[3]**

2. The Gibbs entropy, S, for a system is defined as

$$S = -k \sum_i p_i \ln p_i \,,$$

where the sum is over microstates labelled by the index i, p_i is the probability that the system is in microstate i and k is Boltzmann's constant. Using the method of Lagrange multipliers, maximise the Gibbs entropy subject to certain constraints to show that the canonical ensemble is given by

$$p_i = \frac{1}{Z} \exp\left(-\beta E_i\right) ,$$

where β is the Lagrange multiplier and Z is the canonical partition function, which should be defined. **[6]**

Use the laws of thermodynamics to identify β as

$$\beta = \frac{1}{kT} \,,$$

where T is the temperature. **[5]**

Consider a single quantum anharmonic oscillator, with energy levels given by

$$E_n = \hbar\omega \left[\left(n + \tfrac{1}{2}\right) + \epsilon \left(n + \tfrac{1}{2}\right)^2 \right] , \quad n = 0, 1, 2, \ldots .$$

where ω is the angular frequency of the oscillator and $0 < \epsilon \ll 1$ is a small constant.

(a) Show that to leading order in ϵ, the single particle partition function, $Z_1(\epsilon)$, is given by

$$Z_1(\epsilon) = \frac{c_1}{\sinh(x/2)} \left[1 + \epsilon c_2 x \left(1 + \frac{2}{\sinh^2(x/2)} \right) \right] + O(\epsilon^2) \,,$$

where $x = \beta\hbar\omega$ and c_1 and c_2 are constants to be determined. **[8]**

(b) Furthermore show that the average energy, $\overline{E}$, for a system of N of these uncoupled distinguishable anharmonic oscillators is given by

$$\overline{E} = \frac{1}{2} N\hbar\omega \left[c_3 \coth(x/2) + \epsilon \left(c_4 + \frac{c_5}{\sinh^2(x/2)} \left[1 - x\coth(x/2)\right] \right) \right] + O(\epsilon^2) \,,$$

where again c_3, c_4, c_5 are constants to be determined. **[6]**

3. Droplets of oil, each of mass m, are allowed to fall to the floor in still air at temperature T. Each droplet is released at rest from the same point, at a height h above the floor. The motion of a droplet is treated as Brownian motion, with motion governed by the Langevin equation

$$m\frac{d^2\vec{r}(t)}{dt^2} + \gamma\frac{d\vec{r}(t)}{dt} = \vec{\eta}(t) + m\vec{g}\,,$$

where $\vec{r} = (x, y, z)$ is measured from the droplet point of release, so that x, y are the horizontal coordinates and the vertical coordinate,z, is measured downwards. $\vec{g} = (0, 0, g)$ is the acceleration due to gravity and γ is the friction coefficient. The random force $\vec{\eta}(t) = (\eta_x(t), \eta_y(t), \eta_z(t))$, due to collisions with air molecules is such that it has average values

$$\langle\vec{\eta}(t)\rangle = \vec{0}\,, \qquad \langle\eta_i(t)\eta_j(t')\rangle = \Gamma\delta_{ij}\delta(t - t')\,,$$

where the index $i = x, y$ or z, similarly for the index j, and Γ is a constant.

(a) Show that

$$\frac{d\vec{r}(t)}{dt} = \frac{1}{m}\int_0^t dt'\, e^{(t'-t)/\tau}\left[\vec{\eta}(t') + m\vec{g}\right]\,,$$

giving the relation for τ in terms of m and γ. **[4]**

(b) Hence show that

$$\vec{r}(t) = \frac{1}{m}\int_0^t dt' \int_0^{t'} dt''\, e^{(t''-t')/\tau}\left[\vec{\eta}(t'') + m\vec{g}\right]\,. \qquad \textbf{[1]}$$

(c) Find $\langle z(t)\rangle$. **[4]**

Show for 'early' times that this solution behaves like that of a freely falling particle and for 'late' times the droplet falls with a terminal speed. Make sure to specify the criteria that determine 'early' and 'late' times. **[4]**

(d) Show that

$$\langle x^2(t)\rangle = \frac{\Gamma\tau^2}{m^2}\left[t - \tau(1 - e^{-t/\tau}) - \tfrac{1}{2}\tau(1 - e^{-t/\tau})^2\right]\,. \qquad \textbf{[8]}$$

(e) Given the fluctuation–dissipation theorem for this problem, namely $\Gamma = 2kT\gamma$, estimate the radius $r(h, T)$ of the patch of oil on the floor, as a function of h and T, for 'late' times. **[4]**