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1. (a) Write down Maxwell's equations, and explain how two of these equations lead to a re-expression of the electric and magnetic fields in terms of vector and scalar potentials, $\mathbf{A}$ and ϕ. **[3]**

(b) Explain what is meant by gauge invariance in electromagnetism, and show how it can be used to enforce the Lorentz condition between $\mathbf{A}$ and ϕ. **[3]**

(c) Suppose that the electromagnetic fields are to be described in terms of a superpotential, $\mathbf{Z}$, where $\mathbf{A} = (1/c^2)\dot{\mathbf{Z}}$ and $\phi = -\boldsymbol{\nabla} \cdot \mathbf{Z}$; show that the Lorentz condition is automatically satisfied in this case. If $\mathbf{Z}$ is assumed to satisfy the wave equation $\Box\mathbf{Z} = -(1/\epsilon_0)\,\mathbf{P}$, show that the usual wave equations for $\mathbf{A}$ and ϕ can be obtained, and derive the necessary relations between the vector $\mathbf{P}$ and the charge and current densities. **[6]**

(d) Write down the general solutions to the electromagnetic wave equations for the potentials, and explain how these apply in the case that the source is a charge q, with non-relativistic velocity $\mathbf{v}$. Consider only a distant source in the radiation zone. Explain carefully what is meant by $[\rho]$ and $[\mathbf{j}]$, the retarded values of the charge density and current density, and hence explain why the volume integral of $[\rho]$ only reduces to q in the nonrelativistic limit. **[5]**

(e) A quadrupole source consists of two charges, q, separated by a vector $\mathbf{d}$, with equal and opposite velocities, oscillating with angular frequency ω. Calculate the vector potential in the radiation zone at large radii and show that its amplitude is $\omega\mathbf{n}\cdot\mathbf{d}/c$ times the dipole field of a single charge. Hence obtain the angular distribution of the radiated power from the quadrupole. **[8]**

2. (a) The quantum vector potential can be written as

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}\,\mathbf{e}_{\mathbf{k},\alpha}\left(a_{\mathbf{k},\alpha}(t)e^{i\mathbf{k}\cdot\mathbf{r}} \;+\; a^\dagger_{\mathbf{k},\alpha}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right)\;;$$

write down the meaning of the terms in this expression. **[3]**

(b) If the operators a and $a^\dagger$ for a given mode have the commutator $[a,a^\dagger] = 1$ and the Hamiltonian is $\sum_i(a_i^\dagger a_i + 1/2)\hbar\omega_i$ summed over all modes, prove that the energy eigenvalues are $E = \sum_i(n_i + 1/2)\hbar\omega_i$, where the n_i are independent non-negative integers. **[6]**

(c) The theory of supersymmetry proposes that the photon has a Fermion counterpart that is identical in properties, except that commutators would be replaced by anti-commutators: $aa^\dagger + a^\dagger a = 1$. If eigenstates $|n\rangle$ are defined via $a^\dagger a\,|n\rangle = n\,|n\rangle$, show that n is an integer that obeys the exclusion principle, so that $n = 0$ or $n = 1$ are the only allowed values. **[6]**

(d) If phase in quantum mechanics is to be locally unobservable, show that this requires a first-order perturbation to the Hamiltonian of a charged particle: $\Delta H = -(q/m)\mathbf{A}\cdot\mathbf{p}$ in the gauge where $\phi = 0$. Explain why the full perturbation allows the simultaneous emission or absorption of up to two photons. **[4]**

(e) According to the Golden Rule, the spontaneous decay rate from an excited state $|\mathrm{Y}\rangle$ to a lower state $|\mathrm{X}\rangle$ is

$$\Gamma = \frac{\omega\, d\Omega}{2\pi\hbar c^3 m^2}\frac{e^2}{4\pi\epsilon_0}\;|M_{\mathrm{YX}}|^2 .$$

Write down the matrix element M_{YX} in the dipole approximation and give an expression for the lifetime of the $m = 0$ $2p$ level [wavefunction $(32\pi a^5)^{-1/2} z \exp(-r/2a)$] decaying to the ground state [wavefunction $(\pi a^3)^{-1/2} \exp(-r/a)$]. You may assume that $\int_0^\infty y^4 \exp(-3y/2)\, dy = 256/81$. **[6]**

3. (a) The twice-ionized OIII ion has two outer-shell electrons in the $2p$ state. By factorising the wave function into a function of total spin and a function of total angular momentum, show that the three possible spectroscopic terms are ^3P, ^1D and ^1S. According to Hund's rule, which of these is the ground state, and why? Spectroscopic selection rules for permitted transitions in this case require zero change in spin; thus explain why the transitions between ^1D & ^1S (4363Å) and ^3P & ^1D (4959/5007Å) are both forbidden lines. **[8]**

(b) The above transitions are observed in nebulae as a result of collisional excitation. Discuss the operation of this mechanism, explaining in particular the reciprocity relation obeyed by collisional cross-sections, and the concept of a critical density. Explain why the line emissivity scales in proportion to ion density at high densities, but in proportion to the square of the density at low densities. **[8]**

(c) The de-excitation cross-section affecting the 4959/5007Å transition is 8 times larger than the de-excitation cross-section affecting the 4363Å transition. Hence derive an expression for the ratio of emissivities in these two lines in the limit of low densities. At what temperature are they in the ratio 100:1? **[6]**

(d) Explain how the calculation of part (c) would differ in the case of high densities. In particular, how does the line emissivity ratio now depend on temperature? **[3]**