College of Science and Engineering
School of Physics and Astronomy

Radiation and Matter

SCQF Level 11, U01439, PHY-4-RadMatt

??th May 2016
2:30pm to 4:30pm

Chairman of Examiners
Prof J S Dunlop

External Examiner
Prof C Tadhunter

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1. (a) Maxwell's equations may be written as

$$\nabla \wedge \mathbf{E} = -\dot{\mathbf{B}} \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \nabla \wedge \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \dot{\mathbf{E}}).$$

Show that the electric and magnetic fields can be derived from a vector and a scalar potential as follows:

$$\mathbf{B} = \nabla \wedge \mathbf{A} \qquad\qquad \mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \phi.$$

[5]

(b) Explain what is meant by a gauge transformation of the potentials, and say why such a transformation is possible. Show that a given pair of potentials $(\phi_0, \mathbf{A}_0)$ can always be transformed into new potentials that obey $\nabla \cdot \mathbf{A} + (1/c^2)\partial\phi/\partial t = 0$. **[5]**

(c) In this gauge, the solution for the vector potential can be written as

$$\mathbf{A}(\mathbf{r}, t) = \left(\frac{\mu_0}{4\pi}\right) \int \frac{[\mathbf{j}(\mathbf{r}', t)]}{|\mathbf{r} - \mathbf{r}'|}\, d^3 r'.$$

Explain the meaning of the square brackets inside the integral, and show that the field measured by a sufficiently distant observer satisfies

$$\nabla \wedge \mathbf{A} = -\frac{\mathbf{n}}{c} \wedge \frac{\partial}{\partial t}\mathbf{A},$$

where $\mathbf{n}$ is a unit vector pointing from source to observer. Using one of Maxwell's equations, show that the fields for an oscillating source are transverse, with $\mathbf{E}, \mathbf{B}, \mathbf{n}$ all mutually perpendicular. **[5]**

(d) For a charge q with velocity $\mathbf{v}$, the volume integral of $\mathbf{j}$ is $q\mathbf{v}$. Hence determine the $\mathbf{E}$ and $\mathbf{B}$ fields from an accelerated charge and show that the rate of loss of energy is $\alpha|\dot{\mathbf{v}}|^2$, where α is a constant that need not be explicitly evaluated. **[5]**

(e) A charge is constrained to undergo one-dimensional harmonic motion with angular frequency ω. Discuss conservation of energy and show that the time-averaged radiation emitted can be accounted for if the particle experiences an additional 'radiation reaction' force $\mathbf{F} = \alpha\, d^2\mathbf{v}/dt^2$. **[5]**

2. (a) The classical vector potential can be written as a Fourier superposition:

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} C(\omega)\, \mathbf{e}_{\mathbf{k},\alpha} \left(a_{\mathbf{k},\alpha}(t) e^{i\mathbf{k}\cdot\mathbf{r}} \; + \; a^*_{\mathbf{k},\alpha}(t) e^{-i\mathbf{k}\cdot\mathbf{r}} \right) \;,$$

where $C(\omega)$ is a normalization factor. Explain the meaning of all quantities in this expression. **[5]**

(b) If the electromagnetic field is described with zero scalar potential, compute the contribution of the electric field to the Hamiltonian within a box of volume V, and show that it is

$$H_E = \frac{1}{4} \sum_{\mathbf{k},\alpha} \left(a_{\mathbf{k},\alpha} a^*_{\mathbf{k},\alpha} + a^*_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha} \right) \hbar\omega,$$

provided $|C(\omega)|^2 = \hbar/(2\epsilon_0 V \omega)$. Show that the electric and magnetic energy densities are equal for each mode, so that the total Hamiltonian is $2H_E$. **[5]**

(c) In quantum electrodynamics, the coefficients a, a^* become operators $a, a^\dagger$, with the commutator $[a, a^\dagger] = 1$. Show that $H = \sum_{\text{modes}} (N + 1/2)\hbar\omega$, where the operator $N = a^\dagger a$. Given eigenstates $N\,|n\rangle = n\,|n\rangle$, show that a and $a^\dagger$ act as lowering and raising operators and derive the effect of the operators on the normalization of the states. Prove that n cannot be negative and hence that it must take integer values, $n = 0, 1, 2, \cdots$ **[5]**

(d) According to the Golden Rule, the semiclassical transition rate between two states can be written as

$$\Gamma = \frac{\omega\, d\Omega}{2\pi\hbar c^3 m^2}\; N_{\mathbf{k},\alpha}\; \frac{e^2}{4\pi\epsilon_0}\; \left| M_{\text{YX}}(\mathbf{k},\alpha) \right|^2 .$$

Explain the meaning of the terms in this expression, and say how it is modified in the fully quantum case, especially considering spontaneous transitions. **[5]**

(e) An electron is confined within an infinitely high cubical potential barrier of side L, and is placed in one of the first excited states. Use the dipole approximation to calculate the rate at which the electron makes spontaneous radiative transitions to the ground state (you may assume that $\int_0^\pi \theta \sin\theta \sin 2\theta \; d\theta = -8/9$).

[5]

3. (a) Define specific intensity of radiation, I_ν. For unpolarized radiation, show that specific intensity is related to photon occupation number, N, via

$$I_\nu = (4\pi\hbar\nu^3/c^2)\,N.$$

[5]

(b) Derive the equation of radiative transfer in the form $dI_\nu/d\ell = -\kappa_\nu I_\nu + \mathcal{E}_\nu$. If the radiation passes through material with two non-degenerate energy levels, discuss the contribution of stimulated and spontaneous transitions to the opacity and emissivity. Under what circumstances will I_ν be constant along light rays? **[7]**

(c) In intergalactic space, clouds of Hydrogen are inferred to have a high degree of ionization. Explain why the ionization arises, and why the temperature is therefore expected to be of order $T = 10^4$ K. Calculate the column density of neutral Hydrogen that would yield an optical depth of unity at the line centre for gas at this temperature (assume a Gaussian line profile; central wavelength of Ly α is 121.5 nm; the spontaneous transition rate associated with the Lyman α line is 6.3×10^8 s^{-1}). **[8]**

(d) The spontaneous transition rate associated with the 21-cm line is 2.85×10^{-15} s^{-1}. Calculate the expected peak brightness temperature of 21-cm emission from the cloud in part (c). **[5]**