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Tutorial questions 5

(1) Show that the general expression for a stimulated transition rate can be written as $\Gamma = (\alpha\omega/2\pi m^2 c^2)\, N_\gamma\, d\Omega_\gamma\, |M_{\rm XY}|^2 = \Gamma_s N_\gamma$, where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant. Let ρ denote the number of states, $\rho = V/(2\pi\hbar)^3 d^3p$ in terms of momentum, and justify the following two formulae for cross-sections of bound-free transitions:

(a) Ionization: $\Gamma_s N_\gamma \rho_e = \sigma_{\rm ion} \rho_\gamma N_\gamma c/V$.
(b) Recombination: $\Gamma_s N_e \rho_e = \sigma_{\rm rec} N_e \rho_e (p_e/m)/V$.

Discuss the need for direction averaging of the cross-sections, and show that the direction-averaged results obey the Milne relation $\sigma_{\rm rec}/\sigma_{\rm ion} = (p_\gamma/p_e)^2$ (where the energy relation $p_e^2/2m = p_\gamma c$ should be assumed).

(2) Define the angular momentum operators and show that.

$$[j_x, j_y] = i\hbar j_z, \text{ (cyclic)}; \qquad [j^2, j_z] = 0.$$

Define the raising and lowering operators $j_\pm = j_x \pm i\, j_y$ and deduce their commutator with j_z. Hence show that there are $2j+1$ states, with z-component eigenvalues $m = -j, -(j-1), \cdots, (j-1), j$ in units of $\hbar$. Finally, show that the normalized effect of the operators is $j_\pm |j,\, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j,\, m \pm 1\rangle$.

(3) For the case of two independent angular momenta, j_1 and j_2, the total can range between $j_1 + j_2$ to $|j_1 - j_2|$. Since there must be $(2j_1+1)(2j_2+1)$ states in total, verify this gives the correct count in the cases where the two j's are each either 1/2 or 1.

(4) For the case $j_1 = j_2 = 1/2$, and assuming that the $(S, m) = (1, 1)$ state must factorise as $|\uparrow_1\rangle|\uparrow_2\rangle$, use the total lowering operator to deduce the other m states, and show that they are all symmetric. The total raising and lowering operators must act on the $S = 0$ state to give zero. Hence prove that the singlet state is antisymmetric. Repeat for $j_1 = j_2 = 1$: show that $(L, m) = (2, 1)$ is symmetric, and hence that $(L, m) = (1, 1)$ must be antisymmetric, so that $L = 1$ states are antisymmetric and $L = 0$ is symmetric.

(5) At what density would an HI cloud 30 pc in line-of-sight extent be optically thick ($\tau_{\rm peak} = 1$) if its temperature were 80K?

(6) What are the first few rotational energy levels of CO? (separation of atoms is 1.13Å). Estimate the transition rate $\Gamma_{\rm spon}$ for the $J = 1$ to $J = 0$ rotational transition (permanent dipole/e = 0.0229Å).

(7) Estimate the depth of a cloud (in parsecs) which is just optically thick ($\tau = 1$ at the line centre) in CO $1-0$ radiation. The cloud kinetic temperature is 30 K and the density of CO molecules is 10^6 m^{-3}. Assume the levels are thermally populated.