MPhys Radiation and Matter 2016/17

Tutorial questions 1

(1) Find a prescription for a gauge transformation for $\mathbf{A}$ and ϕ that will make the transform of the expression $\boldsymbol{\nabla}\cdot\mathbf{A}+c^{-2}\partial\phi/\partial t$ equal to zero, if it is not equal to zero for some given set of fields. Show that this Lorentz gauge condition ($\boldsymbol{\nabla}\cdot\mathbf{A}+c^{-2}\partial\phi/\partial t = 0$) is relativistically invariant.

(2) Show that the wave equation in free space (no sources)

$$\nabla^2\phi - \frac{\partial^2\phi}{c^2\partial t^2} = 0$$

has solutions $\phi = \phi(t \pm \mathbf{r}\cdot\mathbf{n}/c)$, and in particular $\exp i(\omega t \pm \mathbf{k}\cdot\mathbf{r})$. Show in these cases that $\omega = kc$. Explain why only one sign choice is normally used. Use Maxwell's equations – without introducing potentials – to show that such an electromagnetic wave has the following properties: (a) $\mathbf{E}, \mathbf{B}, \mathbf{k}$ are mutually perpendicular, and (b) $|\mathbf{E}| = c|\mathbf{B}|$.

(3) In the presence of sources, the electromagnetic potentials can be written in terms of integrals over retarded sources, e.g. $\phi = (1/4\pi\epsilon_0)\int[\rho]/R\, dV$. Explain how this is consistent with the freedom to choose a gauge in which $\phi = 0$.

(4) Show that a free (classical) electron scatters radiation at a rate determined by the Thomson cross-section:

$$\sigma_{\mathrm{T}} = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2$$

(5) A point charge moves non-relativistically in a small circle at constant angular velocity. Find the rate of emission of energy, and compute the decay lifetime of a classical electron in a classical Bohr orbit. Use the analogy $Gm_1m_2 \rightarrow q_1q_2/4\pi\epsilon_0$ to estimate the decay time of the Earth's orbit owing to emission of gravitational radiation.

(6) A non-relativistic particle of mass m and charge q moves with a velocity $\mathbf{v}$ through a uniform magnetic field $\mathbf{B}$. Show that the trajectory is a spiral and derive its radius r and angular frequency ω. Show that electromagnetic radiation is emitted with power

$$P = \frac{q^2}{6\pi c^3\epsilon_0}\left(\frac{vB\cos\theta}{m}\right)^2,$$

where θ is the pitch angle, $\cos\theta = \hat{\mathbf{v}}\cdot\hat{\mathbf{B}}$. As a result, the velocity declines as $v \propto \exp(-t/T)$; give an expression for T.