School of Physics & Astronomy
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Radiation and Matter

PHYS11020 (SCQF Level 11)

Friday 15th May, 2015 14:30 – 16:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **TWO** questions

Special Instructions

- Only the supplied Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied calculators
- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof. J Dunlop
External Examiner: Prof. C Tadhunter

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1. Outline how, starting from the source-free classical wave equation, the vector potential, $\mathbf{A}$, can be quantized and ultimately written in the form

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}\, \mathbf{e}_{\mathbf{k},\alpha} \left[a_{\mathbf{k},\alpha}(t) e^{i\mathbf{k}.\mathbf{r}} \;+\; a^*_{\mathbf{k},\alpha}(t) e^{-i\mathbf{k}.\mathbf{r}} \right]$$

[4]

It can be shown that the operators a and a^* are analogous to the "ladder" operators c, $c^\dagger$, which can be defined in terms of q (the displacement) and p (the corresponding canonical momentum) as follows:

$$c = (2\hbar\omega)^{-1/2}(\omega q + ip) \quad c^\dagger = (2\hbar\omega)^{-1/2}(\omega q - ip).$$

Given the standard value for the commutator between position and momentum, $[q,p] = i\hbar$, and the simple harmonic oscillator Hamiltonian $H = (1/2)(p^2 + \omega^2 q^2)$, show that $[c, c^\dagger] = 1$, and $H = (\hbar\omega/2)(cc^\dagger + c^\dagger c)$. **[4]**

Thus show that c is an "annihilation" operator which, operating on state N, produces a new state of energy $\hbar\omega$ lower, which is amplified by $\sqrt{N}$. Show, similarly, that $c^\dagger$ is a "creation" operator which produces a new state of energy $\hbar\omega$ higher, amplified by $\sqrt{N+1}$. **[6]**

Hence, given the transition rate for photon absorption

$$w(\Omega)_{abs} d\Omega = \frac{\omega d\Omega}{2\pi\hbar c^3 m^2}\, N_{\mathbf{k},\alpha}\, \frac{e^2}{4\pi\epsilon_0}\, |M_{YX}(\mathbf{k},\alpha)|^2$$

write down the analogous expression for photon emission, and explain clearly what $N_{\mathbf{k},\alpha}$ means. **[2]**

Define specific intensity I_ν, and show that the relationship between I_ν and $N_{\mathbf{k},\alpha}$ describing the same field is

$$I_\nu = \frac{2h\nu^3}{c^2} N_{\mathbf{k},\alpha}$$

[5]

The wavelength of Lyman-α emission is 1216Å, and the rate of spontaneous emission is $w_{spon} = 6.3 \times 10^8\,\mathrm{s}^{-1}$. Calculate the specific intensity of the radiation field at which the rate of Lyman-α absorption equals the rate of spontaneous Lyman-α emission. **[4]**

2. Outline, with mathematical illustration, the steps to obtain Fermi's Golden Rule

$$w = \frac{2\pi}{\hbar}|H'_{fi}|^2 \, (dN/dE)_f, \quad E_f = E_i$$

explaining clearly the meaning of $(dN/dE)_f$, and how the requirement for energy conservation arises. **[8]**

When the interaction Hamiltonian for radiation and matter is inserted into the matrix element, the result can be expressed as a sum of two terms, one of which is

$$H'_{fi} = q\sqrt{\frac{\hbar}{2\epsilon_0 \omega V}}\,\sqrt{N_{\mathbf{k},\alpha}}\mathbf{e}_{\mathbf{k},\alpha}.\int \psi_Y^* \, e^{i\mathbf{k}.\mathbf{r}} \, \mathbf{v} \, \psi_X \, dV.$$

Explain how this expression can be simplified further by the dipole approximation, stating clearly when this approximation can be used. **[2]**

Outline why the next "higher-order" term can be subdivided into electric quadrupole and magnetic dipole transitions, and estimate how the rates of such transitions compare with those produced by electric dipole transitions. **[5]**

Explain why, for a single electron transition in a hydrogen atom, the dipole approximation leads to the selection rules $\delta l = \pm 1$, $\delta m_l = 0, \pm 1$. **[4]**

Hence explain and write down the selection rules for the quantum number J describing a multi-electron atom in the case of i) dipole radiation, and ii) electric quadrupole radiation. **[2]**

Why are dipole-forbidden transitions rarely seen in the lab, but so prevalent and important in astrophysics? Give an example of such a transition, and briefly discuss its astrophysical importance. **[4]**

3. The rotational energy levels of the CO molecule have energies proportional to $J(J+1)$ where J is the angular momentum quantum number.

Briefly explain why CO emission is generally only observed for transitions between adjacent rotational energy levels (*i.e.* CO 1–0, CO 2–1, CO 3–2, *etc*) and, given that the emission wavelength of the CO 1–0 transition is 2.6 mm, calculate the wavelengths of the CO 2–1 and CO 3–2 transitions. **[3]**

Define collision cross section σ, collision rate coefficent $\langle \sigma v \rangle$ ($\mathrm{m^3 s^{-1}}$), and critical density n_{crit}. **[3]**

Prove that, in a gas of two-level atoms, with a Maxwellian velocity distribution, the rate coefficient for upward collisions is equal to that for downward collisions multiplied by the Boltzmann ratio of level occupancy. **[3]**

Hence show that the upper level occupancy can be written as

$$n_u = \frac{n_u^{Boltz}}{1 + n_{crit}/n}$$

where

$$n_u^{Boltz} = n_l (g_u/g_l) e^{-\frac{E_u - E_l}{kT}}$$

explaining clearly the meaning of all the terms in this equation. **[4]**

The emission from the $J \to J-1$ transition of CO is proportional to the number of molecules in the upper state. Show that in a cloud where the levels are excited by collisions with molecular hydrogen, the critical density increases monotonically with the level being excited. You may assume the dipole of CO is approximately independent of J, that the cross section is proportional to the dipole squared, and that the spontaneous transition rate is

$$w_{spon} = \frac{4\omega^3}{3\hbar c^3} \frac{e^2}{4\pi\epsilon_0} \, |\mathbf{r}|^2 .$$

[5]

One of the CO emission lines from a molecular cloud is observed to have a full width at half maximum of $1\,\mathrm{kms^{-1}}$ and a peak brightness temperature of 30 K. Assuming the line has a Doppler broadened profile, work out its expected line width if it was optically thin, and hence estimate the optical depth at the centre of the line. **[5]**

What problems does this present for estimating the gas mass of a molecular cloud or a galaxy from CO line emission? **[2]**