College of Science and Engineering
School of Physics & Astronomy

Radiation and Matter
SCQF Level 11, PHYS11020
Thursday, 16th May 2013
9:30am to 11:30am

Chairman of Examiners
Prof J A Peacock

External Examiner
Prof C Tadhunter

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

A sheet of standard constants is available for use during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1. Fermi's Golden Rule is

$$w = \frac{2\pi}{\hbar}|H'_{fi}|^2 \, (dN/dE)_f, \quad E_f = E_i$$

Explain what each term in this equation is. **[3]**

For an atom in initial state X and final state Y the dipole matrix element of H' for absorption of a photon directed along $\mathbf{n}$ within solid angle $d\Omega$ is

$$q\sqrt{\frac{\hbar\omega N_{\mathbf{k},\alpha}}{2\epsilon_0 V}}\,\mathbf{e}_{\mathbf{k},\alpha}.\mathbf{r}_{YX}$$

Explain what the terms in this expression represent. **[3]**

By considering the number density of quantized states in k-space, derive an expression for $(dN/dE)_f$, and hence show that, for all polarizations and angles, the spontaneous dipole transition rate for this system can be written

$$w_{spon} = \frac{4\omega^3}{3\hbar c^3}\,\frac{e^2}{4\pi\epsilon_0}\,|\mathbf{r}_{YX}|^2$$

[6]

By assuming both atomic states X and Y can be described by hydrogenic wavefunctions, and considering the projection of the dipole operator $\mathbf{r}$ in spherical polar coordinates, derive the selection rules for dipole radiation.

$[NOTE: \;\cos\theta P_l^{m_l} = \frac{(l-m_l+1)P_{l+1}^{m_l}+(l+m_l)P_{l-1}^{m_l}}{2l+1}; \quad \sin\theta P_l^{m_l-1} = \frac{P_{l+1}^{m_l}-P_{l-1}^{m_l}}{2l+1})$ **[5]**

Assuming that $\mathbf{r}_{YX}$ is of order the Bohr radius, calculate the transition rate w_{spon} for Lyman-α emission. **[3]**

Explain the implications of this rate for the observability of UV continuum and Lyman-α line emission from galaxies and quasars as viewed at increasingly high redshift back into the reionization epoch. How does the situation differ for the Balmer-α line and why. **[5]**

2. Given the transition rate for photon absorption

$$w(\Omega)_{abs} d\Omega = \frac{\omega d\Omega}{2\pi\hbar c^3 m^2}\, N_{\mathbf{k},\alpha}\, \frac{e^2}{4\pi\epsilon_0}\, |M_{YX}(\mathbf{k},\alpha)|^2$$

write down the analogous expression for photon emission, and explain clearly what $N_{\mathbf{k},\alpha}$ means. **[2]**

Hence write down the corresponding expression for the rate of spontaneous emission, w_{spon}, in the absence of a radiation field. **[1]**

Define specific intensity I_ν, and show that the relationship between I_ν and $N_{\mathbf{k},\alpha}$ describing the same field is

$$I_\nu = \frac{2h\nu^3}{c^2} N_{\mathbf{k},\alpha}$$

[5]

Hence, derive the equation of radiative transfer for a spectral line from a two-state atom

$$\frac{dI_\nu}{dl} = -\kappa_\nu I_\nu + \mathcal{E}_\nu$$

where $\kappa_\nu = (n_l - n_u) w_{spon} (\lambda^2/8\pi)\phi_\nu$, and explain clearly what each term in this equation represents. **[5]**

What happens if **i)** $n_u = n_l$, and **ii)** $n_u > n_l$? **[2]**

Define the optical depth τ_ν and source function S_ν, and show that if κ_ν and S_ν are independent of position, the solution to the equation of radiative transfer is

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

[3]

Show that, if absorption and emission are due to the same process, and the source is in thermal equilibrium and optically thick, that I_ν is given by Planck's Black Body formula. **[3]**

A simplified galaxy model consists of an infinitely extended slab of finite thickness with a uniform mixture of stars and dust. Assuming no background emission, calculate the surface brightness observed as a function of inclination and show that it is independent of inclination in the limit that the amount of dust becomes very large. Specifically, let the slab have a thickness L, and let τ denote the optical depth for a light ray passing perpendicularly through the slab. Assume the emissivity of starlight $\mathcal{E}_\nu$ and the density of dust to be constant throughout the slab. **[3]**

How does the observed flux density from the galaxy depend on inclination in this optically thick situation? **[1]**

3. Show that, if one insists that Schrödinger's equation for a free particle is invariant to local phase changes in the wave function of the form $\psi \rightarrow e^{i\alpha}\psi$ (where $\alpha = \alpha(\mathbf{r}, t)$), one has to introduce a 4-vector potential which obeys the same local gauge invariance transformations as the electromagnetic potential $(\mathbf{A}, \phi)$. **[4]**

Hence show that, adopting the gauge where $\phi = 0$, $\nabla . \mathbf{A} = 0$, the interaction Hamiltonian for radiation and matter has the form (to first order)

$$H' = -(q/m)\mathbf{A}.\mathbf{p}$$

where $\mathbf{A}$ is the electromagnetic vector potential, and $\mathbf{p}$ is the particle momentum. **[3]**

Outline how, starting from the source-free classical wave equation for $\mathbf{A}$, the vector potential $\mathbf{A}$ can be quantized and ultimately written in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k},\alpha} \sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}\, \mathbf{e}_{\mathbf{k},\alpha} \left[a_{\mathbf{k},\alpha}(t)e^{i\mathbf{k}.\mathbf{r}} \;+\; a^*_{\mathbf{k},\alpha}(t)e^{-i\mathbf{k}.\mathbf{r}}\right]$$

and explain what the operators a and a^* do when operating on a radiation state with photon occupation number $N_{\mathbf{k},\alpha}$. **[5]**

Hence justify why, when the interaction Hamiltonian is inserted into the matrix element connecting an initial atomic state X and a final atomic state Y, the result can be expressed as a sum of two terms, one of which is

$$H'_{fi} = q\sqrt{\frac{\hbar}{2\epsilon_0 V\omega}}\, \sqrt{N_{\mathbf{k},\alpha}}\mathbf{e}_{\mathbf{k},\alpha} . \int \psi^*_Y \, e^{i\mathbf{k}.\mathbf{r}}\, \mathbf{v}\, \psi_X \, dV.$$

[2]

By expanding the exponential term in this equation show that, if electric dipole transitions are forbidden, transitions can still be allowed by either the magnetic dipole or electric quadrupole operators, and estimate how the rates for these "higher order" transitions compare with those produced by electric dipole transitions. **[6]**

What kind of transition produces an emission or absorption line at $\lambda = 21$cm from neutral Hydrogen atomic gas? The transition rate w_{spon}(21cm) is only $2.85 \times 10^{-15}\mathrm{s}^{-1}$. Give two reasons why this transition is so unlikely (compared with, say, Lyman-α emission) and explain why the 21cm line is still an important and useful spectral feature in modern astrophysics. **[5]**