College of Science and Engineering
School of Physics & Astronomy

# Radiation and Matter

## SCQF Level 11, PHYS11020

## Thursday 19th May ,2011

## 2:30pm to 4:30pm

**Chairman of Examiners**
Prof J A Peacock

**External Examiner**
Prof C Clarke

Answer **TWO** questions

**The bracketed numbers give an indication of the value assigned to each portion of a question.**

Only the supplied Electronic Calculators may be used during this examination.

A sheet of standard constants is available for use during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

**1.** The Hamiltonian for a simple harmonic oscillator with angular frequency $\omega$ can be written

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

where $q$ is the displacement and $p$ is the corresponding canonical momentum.

Two new operators, $c$, $c^\dagger$, can be defined in terms of $q$ and $p$ as follows:

$$c = (2\hbar\omega)^{-1/2}(\omega q + ip) \quad c^\dagger = (2\hbar\omega)^{-1/2}(\omega q - ip).$$

Given the standard value for the commutator between position and momentum, $[q, p] = i\hbar$, show that $[c, c^\dagger] = 1$, and $H = (\hbar\omega/2)(cc^\dagger + c^\dagger c)$. **[4]**

Hence show that $c$ is an "annihilation" operator which, operating on state $N$, produces a new state of *energy* $\hbar\omega$ lower, which is amplified by $\sqrt{N}$. Show, similarly, that $c^\dagger$ is a "creation" operator which produces a new state of *energy* $\hbar\omega$ higher, amplified by $\sqrt{N+1}$. **[6]**

Briefly outline how these operators can be related to the quantised radiation field as expressed via the electromagnetic vector potential $\mathbf{A}(\mathbf{r}, t)$. **[2]**

Hence, given the transition rate for photon absorption

$$w(\Omega)_{abs} d\Omega = \frac{\omega d\Omega}{2\pi\hbar c^3 m^2} \, N_{\mathbf{k},\alpha} \, \frac{e^2}{4\pi\epsilon_0} \, |M_{YX}(\mathbf{k}, \alpha)|^2$$

write down the analogous expression for photon emission, and explain clearly what $N_{\mathbf{k},\alpha}$ means. **[2]**

By considering a two-level system (with energy gap $\Delta E$) show that, in thermodynamic equilibrium,

$$N_{\mathbf{k},\alpha} = 1/(e^{\Delta E/kT} - 1)$$

and hence deduce Planck's formula for the specific intensity of Black-Body radiation

$$B_\nu(T) = \frac{2\nu^2}{c^2} \frac{h\nu}{e^{h\nu/kT} - 1}.$$

**[5]**

An emission line of $HCO^+$ has been observed to have a full width at half maximum of $0.6\,\mathrm{km\,s^{-1}}$ and a peak brightness temperature of $15\,\mathrm{K}$. Estimate the optical depth assuming the line is optically thick in its centre, and has a Doppler-broadened profile. **[6]**

**2.** Outline, with mathematical illustration, the steps to obtain Fermi's Golden Rule

$$w = \frac{2\pi}{\hbar}|H'_{fi}|^2\,(dN/dE)_f, \quad E_f = E_i$$

explaining clearly the meaning of $(dN/dE)_f$, and how the requirement for energy conservation arises. **[8]**

By insisting that Schrödinger's equation for a free particle is invariant to local phase changes in the wave function of the form $\psi \to e^{i\alpha}\psi$ (where $\alpha = \alpha(\mathbf{r}, t)$), demonstrate that the interaction Hamiltonian for radiation and matter has the form

$$H' = -(q/m)\mathbf{A}.\mathbf{p}$$

where $\mathbf{A}$ is the electromagnetic vector potential, and $\mathbf{p}$ is the particle momentum. **[6]**

When this interaction Hamiltonian is inserted into the matrix element, the result can be expressed as a sum of two terms, one of which is

$$H'_{fi} = q\sqrt{\frac{\hbar}{2\epsilon_0\omega V}}\,\sqrt{N_{\mathbf{k},\alpha}}\mathbf{e}_{\mathbf{k},\alpha}.\int \psi_Y^*\,e^{i\mathbf{k}.\mathbf{r}}\,\mathbf{v}\,\psi_X\,dV.$$

Briefly explain why the matrix element can be written in this form, and then explain how it can be simplified further by the dipole approximation, stating clearly when this approximation can be used. **[6]**

The dipole spontaneous transition rate derived from the above is

$$w_{spon} = \frac{4\omega^3}{3\hbar c^3}\,\frac{e^2}{4\pi\epsilon_0}\,|\mathbf{r}|^2\,.$$

The resulting Lyman-$\alpha$ spontaneous transition rate is $w = 6.3 \times 10^8\,\mathrm{s}^{-1}$. Singly-ionized helium has a structure identical to hydrogen, but scaled differently. By considering a simple Bohr model of the atom, argue that, relative to the H atom, the energy levels in the $He^+$ ion are increased by a factor of 4, while the size scale is reduced by a factor of 2. Hence calculate the spontaneous transition rate for the transition corresponding to Lyman-$\alpha$ in $He^+$. **[5]**

**3.** The rotational energy levels of the CO molecule have energies proportional to $J(J+1)$ where $J$ is the angular momentum quantum number.

Briefly explain why CO emission is generally only observed for transitions between adjacent rotational energy levels (*i.e.* CO 1–0, CO 2–1, CO 3–2, *etc*) and, given that the emission wavelength of the CO 1–0 transition is 2.6 mm, calculate the wavelengths of the CO 2–1 and CO 3–2 transitions. **[3]**

Define collision cross section $\sigma$, collision rate coefficent $\langle \sigma v \rangle$ ($\mathrm{m^3 s^{-1}}$), and critical density $n_{crit}$. **[5]**

Prove that, in a gas of two-level atoms, with a Maxwellian velocity distribution, the rate coefficient for upward collisions is equal to that for downward collisions multiplied by the Boltzmann ratio level of occupancy. **[3]**

Hence show that the upper level occupancy can be written as

$$n_u = \frac{n_u^{Boltz}}{1 + n_{crit}/n}$$

where

$$n_u^{Boltz} = n_l (g_u/g_l) e^{-\frac{E_u - E_l}{kT}}$$

explaining clearly the meaning of all the terms in this equation. **[4]**

The emission from the $J \rightarrow J-1$ transition of CO is proportional to the number of molecules in the upper state. Show that in a cloud where the levels are excited by collisions with molecular hydrogen, the critical density increases monotonically with the level being excited. You may assume the dipole of CO is approximately independent of $J$, that the cross section is proportional to the dipole squared, and that the spontaneous transition rate is

$$w_{spon} = \frac{4\omega^3}{3\hbar c^3} \frac{e^2}{4\pi\epsilon_0} \; |\mathbf{r}|^2 \, .$$

**[5]**

Hence show that the relative intensity of CO lines can be used to determine the hydrogen molecule density in the cloud. **[5]**