College of Science and Engineering
School of Physics and Astronomy

Radiation and Matter

SCQF Level 11, U01439, PHY-4-RadMatt

Thursday, 30th April 2009

9.30 - 11.30 a.m.

Chairman of Examiners
Professor J A Peacock

External Examiner
Professor S Rawlings

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

1. State the Born-Oppenheimer approximation as it applies to small molecules. **[2]**

State the rigid-rotor approximation for the rotational behaviour of the CO molecule and describe how it follows from the Born-Oppenheimer approximation. **[3]**

The rigid-rotor Hamiltonian is

$$H = \frac{\mathbf{J}^2}{2I}$$

where $\mathbf{J}$ is the angular momentum operator and I the moment of inertia. Derive an expression for the reduced mass of the CO molecule. Taking the atomic nuclei of CO to be separated by $R = 1.1$ Å on average, and the atomic masses to be integer multiples of the mass of the hydrogen atom, calculate the reduced mass of $^{12}C^{16}O$ and the moment of inertia. **[4]**

Given that the eigenvalues of $\mathbf{J}^2$ are $J(J+1)\hbar^2$, calculate the frequency of the $J = 1 \rightarrow 0$ transition. **[5]**

What are the wavefunctions appropriate to the rotational states? **[2]**

Given that the permanent electric dipole moment is $|e|\mathbf{d}$, where e is the electronic charge and $\mathbf{d}$ is a vector of length 0.023 Å, calculate the spontaneous emission rate of the $J = 1 \rightarrow 0$ transition from the formula

$$w_{\rm spon} = \frac{4\omega^3}{3\hbar c^3}\frac{e^2}{4\pi\epsilon_0}| < \text{upper } |\mathbf{r}| \text{ lower} > |^2$$

given that the lowest spherical harmonics are **[5]**

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta.$$

Calculate the transition frequency and spontaneous emission rate of the $J = 1 \rightarrow 0$ transition in the next most abundant isotopomer, $^{13}C^{16}O$. **[4]**

2. Given that in the wave zone the vector potential may be written in the general form

$$\mathbf{A}(\mathbf{r},t) = \mathbf{F}(t - \frac{\mathbf{r}.\mathbf{n}}{c})/R$$

where $\mathbf{F}$ is an arbitrary vector-valued function of a single argument (the retarded time), R (assumed constant) is the distance from the centre of the emitting region to the field point at $\mathbf{r}$, and $\mathbf{n}$ is a unit vector in that same direction, derive expressions for $\nabla \wedge \mathbf{F}$ and $\nabla.\mathbf{F}$ in terms of $\frac{\partial \mathbf{F}}{\partial t}$. **[2]**

From Maxwell's equations in empty space

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \wedge \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

and the fact that the scalar potential $\phi = 0$ in the wave zone, show that $\mathbf{n} \wedge \mathbf{E} = c\mathbf{B}$ and $c\mathbf{n} \wedge \mathbf{B} = -\mathbf{E}$. Infer that $\mathbf{E}, \mathbf{B}$ and $\mathbf{n}$ form a right-handed triad like x, y and z. **[2]** **[2]**

Define the Poynting vector $\mathbf{S}$. **[2]**

Show that

$$\mathbf{S} = \epsilon_0 c \dot{\mathbf{A}}_\perp^2 \mathbf{n}$$

[4]

Given the Liénard-Wiechert potential in the wave zone for radiation from a single particle of charge q and velocity $\mathbf{v}$ in the non-relativistic limit

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 q \mathbf{v}}{4\pi R}$$

show that

$$\mathbf{S} = \frac{1}{4\pi R^2} \frac{q^2}{4\pi c^3 \epsilon_0} \dot{v}^2 \sin^2\alpha \, \mathbf{n}$$

[4]

where α is the polar angle. Finally, integrate this over the sphere to obtain Larmor's formula for the power radiated by an accelerated charge

$$P = \frac{q^2 \dot{v}^2}{6\pi c^3 \epsilon_0}$$

[3]

A particle of mass m and charge Q initially moves with a velocity $\mathbf{v}$ perpendicular to a uniform magnetic field of strength $\mathbf{B}_0$. Show that the trajectory is a circle, and derive the radius of the circle r and the angular frequency ω. **[2]**

Show that the particle radiates electromagnetic waves at a rate

$$P = \frac{q^4}{6\pi c^3 \epsilon_0} \left(\frac{vB_0}{m} \right)^2$$

[2]

and that the resulting loss of kinetic energy causes the velocity to decrease as

$$v = v_0 \exp(-t/T)$$

giving an expression for T. **[2]**

3. Define the specific intensity I_ν and give its (MKS) units. **[3]**

Which two aspects of the radiation field does I_ν not encapsulate? **[2]**

The equation of radiative transfer in standard notation is

$$\frac{dI_\nu}{dl} = -\kappa_\nu I_\nu + \epsilon_\nu.$$

What are κ_ν and ϵ_ν called, and what are their units? **[4]**

Show that I_ν is constant in empty space. **[1]**

It follows that I_ν measured by an observer looking at the centre of the solar disc is independent of his distance from the Sun; how is this consistent with the inverse-square law? **[2]**

κ_ν and ϵ_ν for any spectral line in any astrophysical setting both depend on the frequency as some function $\phi(\nu)$. How is this usually normalized? **[1]**

Describe all the processes you can think of that can contribute to the form of $\phi(\nu)$ in various astronomical environments, indicating where appropriate the standard form $\phi(\nu)$ might take if that process were acting alone. **[5]**

Show that these processes fall into two main classes. **[1]**

A uniform interstellar cloud of thickness L along the line of sight absorbs and emits radiation with constant κ_ν and ϵ_ν at a certain frequency ν. Solve for the observed I_ν assuming that no light is incident on the far side of the cloud. **[3]**

How does the solution behave in the limits: a) $L\kappa_\nu << 1$ and b) $L\kappa_\nu >> 1$? **[2]**

How would b) be affected if there were a uniform source on the far side of the cloud? **[1]**