General Relativity
Tutorial 8

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Gravitational waves from black holes

A: Core problems

For weak fields, the metric perturbation $h^{\mu\nu}$ obeys the wave equation

$$\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu},$$

where

$$\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h,$$

$T^{\mu\nu}$ is the energy-momentum tensor, and $\eta^{\mu\nu}$ is the Minkowski metric. The solution for the field is an integral over the retarded source:

$$\bar{h}^{\mu\nu} = -\frac{4G}{c^4} \int \frac{[T^{\mu\nu}]}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'.$$

In the transverse-traceless gauge, only spatial parts of the field are non-zero, and the integral over the source can be expressed using the quadrupole formula:

$$\int T^{ij}\; d^3x = \frac{1}{2}\frac{d^2}{dt^2} \int T^{00} x^i x^j\; d^3x \equiv \ddot{I}^{\,ij}/2$$

(with a diagonal term subtracted to force zero trace).

(1) Apply these formulae to the case of a pair of black holes of masses M_1 and M_2 in a circular orbit with separation a. Consider a single mass M at radius R from the centre of mass, and let the orbit be in the $x-y$ plane: $\mathbf{r} = (R\cos\omega t, R\sin\omega t, 0)$. Since the gravitational wave is transverse, we need axes x' and y' that are perpendicular to the direction of propagation. Assume that these are the usual $\hat{\theta}$ and $\hat{\phi}$ directions in spherical polars, and hence prove that the projected coordinates of the mass are

$$(x', y') = (R\cos\omega t\cos\theta, R\sin\omega t),$$

so that the orbit is seen at an angle θ to face-on.

(2) Hence show that the second time derivative of the inertia tensor is

$$\ddot{I}^{\,ij} = -2MR^2\omega^2 \begin{pmatrix} \cos 2\omega t \cos^2\theta & \sin 2\omega t\cos\theta \\ \sin 2\omega t \cos\theta & -\cos 2\omega t \end{pmatrix}$$

(neglecting the phase shift from retardation).

(3) For masses M_1 and M_2 at radii R_1 and R_2, where $R_1 + R_2 = a$, show that the sum of the effect of the two masses in the inertia tensor can be expressed in terms of the reduced mass, μ:

$$M_1 R_1^2 + M_2 R_2^2 = \mu a^2; \quad \mu \equiv \frac{M_1 M_2}{M_1 + M_2}.$$

The mutual gravitational force is GM_1M_2/a^2; hence deduce Kepler's law:

$$\omega^2 = G(M_1 + M_2)/a^3.$$

(4) Applying trace subtraction, show that the overall gravitational field is

$$h^{ij} = -\frac{4G\mu a^2\omega^2}{c^4 r} \begin{pmatrix} \cos 2\omega t\,(\cos^2\theta + 1)/2 & \sin 2\omega t \cos\theta \\ \sin 2\omega t \cos\theta & -\cos 2\omega t\,(\cos^2\theta + 1)/2 \end{pmatrix}.$$

This field induces relative motion in a pair of test particles:

$$\Delta\ddot{x}_i = \frac{1}{2}\ddot{h}^{ij}\,\Delta x_j.$$

Discuss how a ring of particles is deformed by the passage of the gravitational wave. Explain why the edge-on orbit is said to generate linearly polarized gravitational waves, but the face-on orbit give circularly polarized waves.

B: Further problems

(5) The expression for the gravitational field integrated over the source is of the same form as the integral for the electromagnetic vector potential, $\mathbf{A}$, generated by a charge distribution. For EM radiation, there is an energy density $\sim \epsilon_0 E^2 \sim \epsilon_0 A^2\omega^2$. Show that the dimensions of A/c are mass divided by charge. In the analogy between electromagnetism and gravity, it is common to replace $Q^2/4\pi\epsilon_0$ by GM^2; hence argue that there should be an energy density for gravitational waves of order $U \sim c^2h^2\omega^2/G$. Show that this is consistent with the exact energy loss rate for the binary:

$$-\dot{E} = \frac{32}{5}\frac{G}{c^5}\,\mu^2 a^4\omega^6.$$

(6) The total energy of the binary is $-GM_1M_2/2a$, and the rate of change in this must equal the power given to gravitational waves. Hence show that the orbit shrinks according to

$$a(t)^4 = \frac{256G^3}{5c^5}\,M_1M_2(M_1 + M_2)\,(-t).$$

(7) This weak-field expression will cease to be valid when the two black holes can no longer be treated as point masses. Assume that this stage is reached when the Schwarzschild horizons come into contact, forming a single black hole. According to the Newtonian expression for the binding energy, what fraction of the original mass-energy has been radiated away at this point? What is the frequency of the gravitational radiation at this point of merger? Give a numerical value for the case of LIGO GW150914: $M_1 = M_2 = 30\,M_\odot$ ($1M_\odot \simeq 1.988 \times 10^{30}$ kg).