General Relativity
Tutorial 7

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An exercise in de Sitter Space

(1) A maximally symmetric 4D spacetime is a 4D (hyper)sphere embedded in Euclidean 5D space, with one coordinate chosen to be timelike:

$$x^2 + y^2 + z^2 + w^2 - v^2 = R^2,$$

with the metric

$$ds^2 = -c^2 d\tau^2 = dx^2 + dy^2 + dz^2 + dw^2 - dv^2.$$

Make the substitutions

$$v = \sqrt{R^2 - r^2}\,\sinh(ct/R)$$
$$w = \sqrt{R^2 - r^2}\,\cosh(ct/R),$$

where $r^2 = x^2 + y^2 + z^2$, and show that the metric becomes

$$c^2 d\tau^2 = (1 - r^2/R^2)\, c^2 dt^2 - (1 - r^2/R^2)^{-1} dr^2 - r^2 d\psi^2,$$

where as usual $d\psi$ denotes a 'sky angle': element of length on the surface of a unit sphere.

(2) Consider a source of photons that is stationary at radius r. Show that an observer at the origin receives this radiation with a redshift factor

$$1 + z = (1 - r^2/R^2)^{-1/2},$$

and hence that to lowest order there is a quadratic distance-redshift relation: $z \simeq r^2/2R^2$.

(3) Use the Euler-Lagrange approach to derive the equations for a radial geodesic of a massive particle in this spacetime:

$$\dot{t} = k/(1 - r^2/R^2)$$
$$\dot{r} = c\,\sqrt{k^2 - 1 + r^2/R^2},$$

where k is a constant. What is the physical significance of k?

(4) A particle is released from rest at $r = r_i$ at proper time $\tau = \tau_i$. What is the constant k in this case? Show that the geodesic equation of motion is solved by

$$\Delta\tau \equiv \tau - \tau_i = (R/c)\ln\left(r/r_i + \sqrt{(r/r_i)^2 - 1}\right),$$

and hence that

$$r/r_i = \cosh(c\Delta\tau/R),$$

so that a free particle will not remain at constant r.

(5) Repeat the calculation of part (2) for photons released by the particle in part (4). First argue that the relation between changes in received and emitted time is

$$c\delta t_r = c\delta t_e + \frac{\delta r_e}{1 - r_e^2/R^2}.$$

Using the expression for $\dot{t}$ and the result from part (4), show that

$$\frac{\delta\tau_r}{\delta\tau_e} = \frac{\sqrt{1 - r_i^2/R^2}}{1 - r_e^2/R^2} + \frac{\sqrt{r_e^2/R^2 - r_i^2/R^2}}{1 - r_e^2/R^2}.$$

For $r_i \ll r_e \ll R$, show that the distance-redshift relation is linear to lowest order (a result first derived by Weyl in 1923, and which was well-known to Hubble before his 'discovery' of a linear distance-redshift relation):

$$z \simeq r_e/R.$$

(6) The static form of the de Sitter metric is similar to the Schwarzschild metric:

$$c^2 d\tau^2 = A(r)c^2 dt^2 - A(r)^{-1} dr^2 + r^2 d\psi^2.$$

In applying the field equations to the Schwarzschild metric in tutorial 6, we derived the components of the Ricci tensor for a metric of this form:

$$\begin{aligned} R^t_t = R^r_r &= -A''/2 - A'/r \\ R^\theta_\theta = R^\phi_\phi &= -A'/r + (1 - A)/r^2, \end{aligned}$$

where dashes denote d/dr. Show that the field equations for a universe containing only a non-zero cosmological constant, Λ, are $R^\mu{}_\nu = \Lambda g^\mu{}_\nu$, and hence show that the de Sitter metric is a solution of the field equations in such a universe. Give the relation between Λ and the curvature radius, R.

(7) Define comoving radius, r', using the asymptotic exponential motion of a test particle: $r = r' \exp(ct'/R)$, where t' is a new time coordinate that is equal to the proper time of the moving particle. Write down an expression for dr in terms of dr' and dt'. In general, $dt = A\, dt' + B\, dr'$. Since t' is the proper time for a particle, we require $d\tau = dt'$ if $dr' = 0$. If we also require that there be no cross terms $\propto dt'\, dr'$, show that this fixes A and B, and hence that the de Sitter metric can be cast into the expanding form

$$c^2 d\tau^2 = c^2 dt'^2 - e^{2ct'/R}(dr'^2 + r'^2 d\psi^2).$$

Use the Euler-Lagrange equations to verify that a free particle can remain at a fixed value of r' indefinitely.