General Relativity

Tutorial 6

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The Schwarzschild Metric

In this Tutorial we will derive the Schwarzschild metric, using symmetry principles and solving the Einstein Equations.

(1) Argue carefully that one can choose coordinates such that the metric of a static, spherically symmetric spacetime may be written

$$c^2 d\tau^2 = A(r)c^2 dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

and explain precisely the meanings of the coordinates r and t.

(2) From the Euler-Lagrange equations, or otherwise, compute the Affine connections, and show that there are 9 non-zero elements (plus 4 symmetric counterparts):

$$\begin{aligned}
&\Gamma^0{}_{01} = A'/2A; \\
&\Gamma^1{}_{00} = A'/2B; \quad \Gamma^1{}_{11} = B'/2B; \quad \Gamma^1{}_{22} = -r/B; \quad \Gamma^1{}_{33} = -r\sin^2\theta/B; \\
&\Gamma^2{}_{12} = 1/r; \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta; \\
&\Gamma^3{}_{13} = 1/r; \quad \Gamma^3{}_{23} = \cos\theta/\sin\theta
\end{aligned} \tag{2}$$

(3) Using the definition

$$R_{\mu\nu} = \partial_\nu \Gamma^\alpha{}_{\mu\alpha} - \partial_\alpha \Gamma^\alpha{}_{\mu\nu} + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} - \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\mu\nu}, \tag{3}$$

compute the diagonal components of non-zero of the Ricci tensor:

$$\begin{aligned}
R_{00} &= -(A'/2B)' + (A'/2A - B'/2B - 2/r)(A'/2B) \\
R_{11} &= (A'/2A)' + (A'/2A)^2 - (A'/2A + 2/r)(B'/2B) \\
R_{22} &= -1 + (r/B)' + (r/B)(A'/2A + B'/2B) \\
R_{33} &= R_{22}\sin^2\theta
\end{aligned} \tag{4}$$

(the off-diagonal elements vanish identically, but we haven't yet reached the stage of the course where we can prove this easily).

The above manipulation is straightforward in principle, but will cover 2-3 pages. Have the list of connections easily to hand, and use it to spot which items in the sums inside $R_{\mu\nu}$ gain non-zero contributions. For example,

$$R_{00} = 0 - \partial_\alpha \Gamma^\alpha{}_{00} + \Gamma^\alpha{}_{0\beta}\Gamma^\beta{}_{0\alpha} - \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{00}, \tag{5}$$

where the first 0 is because nothing is time dependent. In the first product of Γ's, α and β must both be either 1 or 0 just based on the allowed combinations in the lower indices. In the second product, β must be 1 based on the lower 00 indices. Therefore

$$R_{00} = \partial_\alpha \Gamma^\alpha{}_{00} + 2\Gamma^0{}_{01}\Gamma^1{}_{00} - \Gamma^1{}_{00}\left(\Gamma^0{}_{01} + \Gamma^1{}_{11} + \Gamma^2{}_{21} + \Gamma^3{}_{31}\right), \tag{6}$$

where symmetry of $\alpha\Gamma^\alpha{}_{\mu\nu}$ in the lower indices has been used often. For the other Ricci elements, some of the sums are longer. An effective strategy is to write out the sum on one index explicitly, and then think about which terms in the 4 sub-sums survive, e.g. for R_{11} you will need

$$\Gamma^\alpha{}_{1\beta}\Gamma^\beta{}_{1\alpha} = \Gamma^\alpha{}_{10}\Gamma^0{}_{1\alpha} + \Gamma^\alpha{}_{11}\Gamma^1{}_{1\alpha} + \Gamma^\alpha{}_{12}\Gamma^2{}_{1\alpha} + \Gamma^\alpha{}_{13}\Gamma^3{}_{1\alpha}. \tag{7}$$

Which values of α in each sum give a non-zero term?

[PTO]

This by-hand approach is probably the fastest way to tackle a simple case, but even so it is hardly a thing of beauty. A general alternative might be to use the matrix method mentioned in the lectures, which the tutors can show you, although this generates the entire Riemann tensor, which is more than you need. The amount of careful work needed is still considerable, but it is at least more obvious how things could generalise to a more complicated case.

Having got to the end of the derivation, you may like to consult Schwarzschild's original 1916 paper, which can be found in translation at `http://arxiv.org/abs/physics/9905030`. You will impressed to find that he uses much less space, on account of two critical simplifications: (1) He assumes a metric with $\det(g^{\mu\nu}) = -1$ right from the start, although this isn't given much justification. But with this assumption, the expression for the Ricci tensor is simplified; (2) He works in the equatorial plane, setting $\sin\theta = 1$. His argument for being allowed to do so is that the Ricci tensor only differentiates the connections once, and derivatives of $\sin^2\theta$ are zero at $\theta = \pi/2$. Our connections also contain combinations like $\sin\theta\cos\theta$, whose derivatives don't vanish, so it seems like working in the equator isn't allowed. But Schwarzschild also used the further convenient trick of using $\cos\theta$ as a coordinate rather than θ. So if $w \equiv \cos\theta$, the angular part of the line element changes from $r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ to $r^2 dw^2/(1-w^2) + r^2(1-w^2)\, d\phi^2$, and in these terms the connection coefficients only contain the combination $1 - w^2$, so setting $w = 0$ causes no problems with the derivatives. You may prefer to adopt Schwarzschild's simplification (2) in deriving the Ricci tensor.

(4) By a suitable combination of $R_{00} = 0$ and $R_{11} = 0$, prove that $A'B + B'A = 0$, and hence that $A \propto 1/B$.

(5) Combining this with $R_{22} = 0$, prove that $(r/B)' = 1$ and hence that

$$B = (1 + \beta/r)^{-1}, \tag{8}$$

where β is a constant of integration.

(6) Check that these solutions for A and B satisfy $R_{00} = 0$ and $R_{11} = 0$ individually.

(7) Thus $A = \alpha(1 + \beta/r)$. How are the two unknowns α and β determined?