General Relativity

Tutorial 5

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A: Core problems

(1) Consider the surface of a cylinder, using polar coordinates (r, ϕ, z). All components of the affine connection vanish, and hence the surface of a cylinder is not curved. Prove this in two ways: (a) directly from derivatives of the metric; (b) using the Euler–Lagrange approach, with arc-length $d\ell = (r^2 d\phi^2 + dz^2)^{1/2}$ as affine parameter.

(2) Consider the operation of parallel transport on the 2D curved surface of a sphere of radius R, embedded in 3D Euclidean space.

(a) Using the 3D position vector in polar coordinates, $\mathbf{r} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, obtain the 2D basis vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. Show that these are orthogonal but not orthonormal.

(b) What are the components of the metrics $g^{\mu\nu}$ and $g_{\mu\nu}$ on the surface of the sphere?

(c) Compute the components of the Affine connections directly from the metric, and check that these match those deduced from the Euler–Lagrange equations.

(d) Define a tangent plane centred at $\theta = \theta_T$, $\phi = \phi_T = 0$ with Cartesian basis vectors $\hat{\mathbf{x}} = \hat{\mathbf{e}}_\phi$, $\hat{\mathbf{y}} = -\hat{\mathbf{e}}_\theta$ at the tangent point (note the hats, denoting normalization of the basis vectors). Show that the 2D components of the basis vectors projected onto this plane are

$$\begin{aligned} \mathbf{e}_\theta &= R(\cos\theta\sin\phi, -[\sin\theta\sin\theta_T + \cos\theta\cos\theta_T\cos\phi]) \\ \mathbf{e}_\phi &= R(\sin\theta\cos\phi, \sin\theta\cos\theta_T\sin\phi). \end{aligned} \tag{1}$$

In the tangent plane, covariant derivatives of these vectors should be the same as coordinate derivatives, so $\nabla_i \mathbf{e}_j = \partial_i \mathbf{e}_j = \Gamma^k_{ij}\mathbf{e}_k$ (i.e. differentiating the above 2D forms of the basis vectors projected into the tangent plane with respect to θ and ϕ). Hence obtain the components of the affine connection for the spherical manifold and verify that they agree with those obtained earlier.

(e) Show that changing the θ coordinate does not rotate the parallel-transported basis vectors, but that a change in ϕ does cause rotation (hint: consider the projection of the unit basis vectors, and ask how these change at different points on the tangent plane). Hence show that parallel transport of a vector around a non-great circle at constant θ causes rotation by an angle $2\pi\cos\theta$ clockwise on the xy plane – see the animation at `https://www.youtube.com/watch?v=8gjm8u-PpsY`.

B: Further problems

(3) Let g denote the determinant of $g_{\mu\nu}$, viewed as a matrix (note that g will be negative). The variation in the determinant can be deduced by using $\det(AB) = \det A \det B$, so that $\det(A + \delta A) = \det A \det(A^{-1}[A + \delta A])$. Argue that the determinant of a matrix that is close to the identity is 1 plus the trace of the perturbation to the matrix (to first order in the perturbation) to show that this is $\delta \det A = \det A \,\mathrm{Tr}(A^{-1}\delta A)$, and thus that

$$\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu}. \tag{2}$$

Use this relation to show that

$$\Gamma^{\mu}{}_{\mu\nu} = \partial_\nu \ln |g|^{1/2}. \tag{3}$$

(hint: use the definition of Γ to write an expression for $\partial_\alpha g_{\mu\nu}$. We actually started from this expression in deriving the relation between Γ and the metric – see p12 of the notes). Hence show that the covariant generalization of the divergence of a vector is

$$\nabla_\mu V^\mu = |g|^{-1/2}\partial_\mu \left(|g|^{1/2} V^\mu\right), \tag{4}$$

and that the covariant generalization of the Laplacian is

$$\nabla_\mu \nabla^\mu \phi = |g|^{-1/2}\partial_\mu \left(|g|^{1/2} g^{\mu\nu}\partial_\nu \phi\right). \tag{5}$$