General Relativity

Tutorial 4

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A: Core problems

(1) A 4-vector has a coordinate transformation $V'^{\mu} = (\partial x'^{\mu}/\partial x^{\nu})\, V^{\nu}$. If we define a covector via $U_{\mu} V^{\mu} =$ invariant, derive the coordinate transformation for U_{μ}.

Use the same reasoning with $T_{\mu\nu} U^{\mu} V^{\nu} =$ invariant to derive the transformation law for a rank-2 cotensor $T_{\mu\nu}$.

(2) Show that the Kronecker delta obeys the transformation law expected for a mixed tensor:

$$\delta'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \delta^{\alpha}{}_{\beta}.$$

What is the value of $\delta^{\mu}{}_{\mu}$?

(3) The matrix $g^{\mu\nu}$ is *defined* as the inverse of the metric tensor $g_{\mu\nu}$. Given the transformation law for $g_{\mu\nu}$, prove that $g^{\mu\nu}$ is a vectorial tensor. Hint: consider

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} g'_{\mu\nu},$$

and show it is equal to δ^{λ}_{ν}.

(4) Given the definition of the affine connection in terms of the metric and its derivatives,

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right),$$

Prove that the metric tensor has vanishing covariant derivative:

$$\nabla_{\lambda} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}{}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}{}_{\lambda\nu} g_{\mu\rho} = 0.$$

B: Further problems

(1) Recast the above coordinate transformations in matrix notation: If $\mathbf{V}^{\bullet\prime} = \mathbf{M}\cdot\mathbf{V}^{\bullet}$ and $\mathbf{V}_{\bullet}{}' = \mathbf{N}\cdot\mathbf{V}_{\bullet}$, prove that $\mathbf{N} = (\mathbf{M}^{T})^{-1} = (\mathbf{M}^{-1})^{T}$ [here we use $\mathbf{V}^{\bullet}$ and $\mathbf{V}_{\bullet}$ to denote respectively the contravariant and covariant vectors]. What are the corresponding equations for the transformation of tensors, viewed as matrices? Similarly, prove that the inverse of the covariant metric tensor has the matrix transformation of a contravariant tensor.

(2) Show that the affine connection transforms according to

$$\Gamma'^{\lambda}{}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \Gamma^{\sigma}{}_{\kappa\rho} + \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}}.$$

Hint: start with the definition $\Gamma^{\lambda}{}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$, in terms of LIF coordinates ξ^{μ}.

Hence show that the covariant derivative of a vector is a tensor. Hint: the apparent extra terms that appear cancel out. This can be shown by considering that

$$\frac{\partial x'^{\kappa}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\kappa}_{\mu},$$

whose derivative with respect to x' (in particular) is zero.

(3) The Riemann tensor is

$$R^{\alpha}{}_{\sigma\rho\beta} \equiv \partial_{\rho}\Gamma^{\alpha}{}_{\beta\sigma} - \partial_{\beta}\Gamma^{\alpha}{}_{\rho\sigma} + \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\nu}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\nu}{}_{\sigma\rho}.$$

What is the Riemann tensor in a local inertial frame? Show that in this frame

$$\partial_{\mu}R^{\alpha}{}_{\beta\gamma\delta} + \partial_{\gamma}R^{\alpha}{}_{\beta\delta\mu} + \partial_{\delta}R^{\alpha}{}_{\beta\mu\gamma} = 0.$$

Hence argue that in general

$$\nabla_{\mu}R^{\alpha}{}_{\beta\gamma\delta} + \nabla_{\gamma}R^{\alpha}{}_{\beta\delta\mu} + \nabla_{\delta}R^{\alpha}{}_{\beta\mu\gamma} = 0.$$

Defining the Ricci tensor and scalar as $R_{\alpha\beta} = R^{\mu}{}_{\alpha\beta\mu}$ and $R \equiv R^{\mu}{}_{\mu}$, prove that

$$\nabla_{\mu}R - 2\nabla_{\alpha}R^{\alpha}{}_{\mu} = 0$$

(you may assume the symmetry $R^{\alpha\beta}{}_{\beta\mu} = -R^{\beta\alpha}{}_{\beta\mu}$).

Hence prove that the covariant divergence of the Einstein tensor vanishes:

$$\nabla_{\beta}G^{\alpha\beta} = 0; \qquad G^{\alpha\beta} \equiv R^{\alpha\beta} - (1/2)Rg^{\alpha\beta}.$$