General Relativity
Tutorial 3

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A: Core problems

(1) The Schwarzschild metric outside a compact, spherically symmetric mass is given by

$$c^2 d\tau^2 = \alpha c^2\, dt^2 - \alpha^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

where $\alpha(r) \equiv 1 - 2GM/(rc^2)$, τ is the invariant proper time for some particle, and the co-ordinate system on the RHS is defined for an observer at infinity.

(a) Show that for orbits in the equatorial plane, two constants of the motion are $\alpha\dot{t} = k$ and $r^2\dot{\phi} = h$. What does the dot denote?

(b) Show that, for a particle with non-zero rest mass,

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} - \frac{2GMh^2}{r^3c^2} = c^2(k^2 - 1) = \text{constant}.$$

Hence, writing $H \equiv h/(R_g c)$, and $R \equiv r/R_g$, where $R_g \equiv GM/c^2$ is the 'gravitational radius', show that the effective potential is

$$\Phi/c^2 = \frac{H^2}{2R^2} - \frac{1}{R} - \frac{H^2}{R^3}.$$

(c) Find the value of H for which unbound orbits disappear (as H is reduced), and calculate the radius R at which the effective potential peaks.

(d) Sketch the effective potential for this value of H.

(e) For this value of H, find the radius R of stable circular orbits of massive particles.

(f) Calculate the speed of the orbit as measured by a stationary observer at that radius.

B: Further problems

(2) For a particle moving in the Schwarzschild metric:

(a) Consider a stationary particle with radial coordinate $r > 2GM/c^2$. How is its time related to the coordinate time t?

(b) A particle orbits in a circular (equatorial) orbit at radius r. If the rate of change of ϕ with respect to the particle's proper time τ is $d\phi/d\tau$, calculate the relative rate of time passage of τ and t.

(c) A particle orbits, without loss of generality, in the plane $\theta = \pi/2$. By considering the Euler-Lagrange equations, or otherwise, show that $u \equiv 1/r$ for the orbit satisfies

$$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2k^2}{h^2} - \alpha\left(\frac{c^2}{h^2} + u^2\right) \qquad (1)$$

and identify the constants h and k.

(d) An object is projected from a point at $r = a > 2GM/c^2$ with a speed v (as measured by a *local* observer at rest) perpendicular to the radial direction. Show that

$$k = \sqrt{\frac{\alpha}{1 - v^2/c^2}}; \qquad h = \frac{rv}{\sqrt{1 - v^2/c^2}}. \tag{2}$$

Hint: you need to consider 3 time coordinates in this problem.

(e) Consider the form of the equation for $(du/d\phi)^2$, in particular the solutions where it is zero (other than $1/a$). Show that the orbit spirals in to the centre if

$$v < \frac{2c}{1 + \frac{c^2 a}{2GM}}. \tag{3}$$

Hint: Consider when roots of $(du/d\phi)^2 = 0$ coincide.