General Relativity

Tutorial 2

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A: Core problems

1. (i) By considering the Euler-Lagrange equations (ELII), compute the affine connections for the SR Minkowski metric in spherical polars.
   (ii) Compute the ${\Gamma^\phi}_{r\phi}$ connection directly, by using the general expression:
   $$ {\Gamma^\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right] $$

2. From the equations of motion obtained in question 1, show that the orbit of a free particle in the equatorial plane obeys
   $$ r^2 \frac{d\phi}{dt} = h = \text{constant (specific angular momentum)} $$
   and
   $$ \left(\frac{dr}{dt} \right)^2 + \frac{h^2}{r^2} = \text{constant (energy equation).} $$

3. Consider the Minkowski spacetime metric for cylindrical polars (ct, ρ, ϕ, z), and show that a massive particle in an orbit with angular speed $d\phi/d\tau = \Omega$ feels an inertial force
   $$ \frac{d^2\rho}{d\tau^2} = \rho\Omega^2. $$
   Derive this from the Geodesic Equation by calculating the affine connection, Γ, and then by the Euler-Lagrange Equations.

B: Further problems

1. Consider the line element of 2D Euclidean space in polar coordinates:
   $$ ds^2 = dr^2 + r^2 d\theta^2. $$
   Using the Lagrangian approach, or otherwise, obtain the geodesic equation for this space and show that this implies the following equation:
   $$ (dr/d\theta)^2 + r^2 = kr^4, $$
   where k is a constant. Write down the equation of a straight line in this space, and show that it satisfies the above relation.

2. If a metric is diagonal, show that the affine connection $\Gamma^\alpha_{\mu\nu}$ vanishes unless at least two of its indices are equal, and that the only non-zero terms are
   $$ \Gamma^\alpha_{\beta\beta} = -\frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{\beta\beta}}{\partial x^\alpha}; \quad \Gamma^\alpha_{\alpha\beta} = \frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{\alpha\alpha}}{\partial x^\beta}; \quad \Gamma^\alpha_{\alpha\alpha} = \frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{\alpha\alpha}}{\partial x^\alpha}, $$
   where in this instance repetition of indices does not imply summation, and α and β are distinct.

3. From the Euler-Lagrange equations, prove that the statement that τ_{AB} is stationary is equivalent to the geodesic equation,
   $$ \frac{d^2 x^\lambda}{d\tau^2} + {\Gamma^\lambda}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. $$
   (Take $p = \tau$). You will need to pay attention to which quantities depend on x^μ only, and will need the chain rule to change some $d/d\tau$ terms to partial derivatives. You will also need to use the symmetry of the metric tensor.