General Relativity
Tutorial 1

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A: Core problems

1. Starting from the usual Lorentz transformation for an observer moving with velocity v along the x axis, verify that the norm of a spacetime interval, $dx^\mu dx_\mu$ is indeed invariant.

2. Define the 4-velocity and 4-acceleration in SR as $U^\mu = dx^\mu/d\tau$ and $A^\mu = dU^\mu/d\tau$. The components of the 4-velocity for a particle with velocity $\mathbf{v}$ are $U^\mu = (\gamma c, \gamma \mathbf{v})$; what are the components of A^μ?. By working in the rest frame, show that the invariant $A^\mu U_\mu$ vanishes.

3. Consider a point on the surface of a sphere of radius R. Write down its Cartesian coordinates in terms of the polar coordinates θ and ϕ. By differentiating this expression, or otherwise, show that the element of length on the surface of the sphere, $d\ell$, satisfies $d\ell^2 = g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$ and give expressions for the two metric components $g_{\theta\theta}$ and $g_{\phi\phi}$.

4. Write down the metric tensor $g_{\mu\nu}$, and its inverse $g^{\mu\nu}$, and the determinant $g \equiv \det g_{\mu\nu}$ for the Minkowski coordinate systems of Special Relativity,

   - $c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$ (Cartesian)
   - $c^2 d\tau^2 = c^2 dt^2 - (d\rho^2 + \rho^2 d\phi^2 + dz^2)$ (Cylindrical polars)
   - $c^2 d\tau^2 = c^2 dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$ (Spherical polars)

   Compare the components of the 4-vectors dx^μ and $dx_\mu \equiv g_{\mu\nu} dx^\nu$ in these cases.

5. The twin paradox involves twins A and B, each equipped with a clock. A remains on Earth, while B travels a distance d on a rocket at velocity v, fires the engines briefly to reverse the rocket's velocity, and returns. According to an SR analysis by A, B's clock will indicate a shorter time for the journey than A's: $t_{\rm B} = \gamma^{-1}\, t_{\rm A}$, or a time difference of $\Delta t \simeq (2d/v)(v/c)^2/2$. Use the concept of gravitational time dilation to show (to lowest order in v) that this is also the time difference calculated by B in their non-inertial frame.

B: Further problems

1. Consider two particles with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. By using the invariant combination of their 4-velocities, $U^\mu V_\mu$, show that the Lorentz factor of their relative motion is $\gamma_{12} = \gamma_1\gamma_2(1 - \mathbf{v}_1 \cdot \mathbf{v}_2/c^2)$. For the case where the velocities are parallel, show that this reduces to the usual velocity addition law $v_{12} = (v_1 - v_2)/(1 - v_1 v_2/c^2)$.

2. Similarly to the above analysis of the twin paradox, use gravitational time dilation to explain why stars suffer no time dilation due to their apparently high transverse velocities as viewed from the frame of the rotating Earth.

3. Compute the metric tensor for flat space described by coordinates (ct, x', y', z) where $x = x' + y'$ and $y = x'y'$, and (ct, x, y, z) are the usual Minkowski coordinates.

4. By a suitable coordinate transformation, show that the line element

$$c^2 d\tau^2 = (c^2 - a^2t^2)dt^2 - 2at\, dtdx - dx^2 - dy^2 - dz^2,$$

where a is a constant, can be reduced to the Minkowski line element [hint: focus on the unusual cross term, and think how it might arise from an expression of the form $(A\, dx + B\, dt)^2$].

5. The geodesic equation for the motion of free particles in Minkowski spacetime is

$$\ddot{x}^\mu = 0,$$

where $\dot{} \equiv d/d\tau$. Solve the time and spatial equations to show that these particles move along straight worldlines in the Minkowski spacetime coordinates.

6. The line element for a particle in Minkowski spacetime in spherical coordinates is

$$c^2 d\tau^2 = c^2 dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2).$$

Using the affine connection,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\eta}(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\eta\mu} - \partial_\eta g_{\mu\nu}),$$

find $\Gamma^r{}_{\theta\theta}$ and $\Gamma^r{}_{\phi\phi}$. Hence show the radial equation of motion is

$$\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0,$$

explaining why the other affine connections in this equation are zero. Give a physical interpretation of this equation.

7. The line element for a satellite moving in the gravitational field, in a circular orbit in the plane $\theta = \pi/2$, around the Earth, with mass M, is

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - r^2 d\phi^2.$$

Using Kepler's 3rd Law for orbits, show that the proper time for the particle, τ, and the coordinate time, t, are related to first order by

$$\frac{d\tau}{dt} = 1 - \frac{3GM}{2rc^2}.$$

What is the same relation for a receiver dish fixed on the surface of the Earth at the equator? Hence what is the time difference between pulses of light sent radially from the satellite to the dish, and how might this affect the Global Positioning System?