School of Physics & Astronomy

General Relativity

PHYS11010 (SCQF Level 11)

Wednesday 0th May, 2025 13:00 - 15:00
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **BOTH** questions

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous barcodes to *each* script book used.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairperson of Examiners: Prof. K Rice
External Examiner: Prof. I McCarthy

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1.

(a) Explain what is meant by the terms 'Local Inertial Frame', 'Weak & Strong Equivalence Principle', and say how these concepts allow us to deduce that spacetime is a Riemannian manifold with a metric structure. Write down the coordinate transformation for the metric tensor in both its contravariant and covariant forms, and hence show from the equivalence principle that it must be symmetric. **[5]**

(b) Trajectories of free particles obey an action principle whereby $\int L^2 \, d\lambda$ is stationary with respect to variations in the path $x^\mu(\Lambda)$, and

$$L^2 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda},$$

where λ is an affine parameter. Explain why this principle is equivalent to saying that massive particles travel on trajectories of extremal proper time, and say how the principle can be used to derive the equations of motion. **[5]**

(c) Write down the Schwarzschild metric, and hence derive the equations of motion for a massive particle that orbits in the equatorial plane with polar angle $\theta = \pi/2$. How do the equations change in the case of a photon? Show that there is a single radius for the 'photon sphere' at which light can execute a circular orbit, and derive whether or not this orbit is stable. **[5]**

(d) A volunteer is released from rest at a radial coordinate distance $r_0 > 2GM/c^2$ above a black hole of mass M. Solve the radial equation of motion and show that the observer crosses the event horizon and reaches $r = 0$ after a finite time as measured by their clock [The integral $\int_0^1 (1/x - 1)^{-1/2} \, dx = \pi/2$ may be assumed]. If the event horizon is thought of as a physical surface, how fast does this surface appear to be moving towards the falling observer as they meet it? [hint: the relative Lorentz factor of two bodies is $U^\mu V_\mu / c^2$, where U^μ and V^μ are their 4-velocities] **[5]**

(e) A photon is projected towards a a Schwarzschild black hole of mass M, with 'impact parameter' b (travelling parallel to a line through the centre of the black hole, but offset by a distance b). Show that for a critical value of b the photon will be captured into a circular orbit at $r = r_\gamma$. Since photon trajectories are invariant under time reversal, show that the apparent angular size of the photon sphere as observed by the Event Horizon Telescope is increased by a factor $\sqrt{3}$ over the naive expectation of r_γ / D, where D is the distance to the black hole (large compared to the horizon size). **[5]**

2.

(a) The affine connection is related to the metric tensor as follows:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right).$$

Consider the near-Newtonian situation where the metric is independent of time and is close to that of special relativity. Write down the geodesic equation of motion and show that to lowest order the acceleration of a stationary free particle is proportional to the gradient of g_{00}. **[7]**

(b) The Riemann tensor is

$$R^{\mu}{}_{\alpha\beta\gamma} \equiv \partial_{\beta}\Gamma^{\mu}{}_{\gamma\alpha} - \partial_{\gamma}\Gamma^{\mu}{}_{\beta\alpha} + \Gamma^{\mu}{}_{\beta\nu}\Gamma^{\nu}{}_{\alpha\gamma} - \Gamma^{\mu}{}_{\gamma\nu}\Gamma^{\nu}{}_{\alpha\beta}\,.$$

Again considering the near-Newtonian situation of part (a), show that the Ricci tensor can be approximated as follows:

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\beta\mu} \simeq \frac{1}{2}\eta^{\mu\nu}\left\{\partial_{\alpha}\partial_{\beta}g_{\mu\nu} - \partial_{\nu}\partial_{\beta}g_{\mu\alpha} - \partial_{\alpha}\partial_{\mu}g_{\beta\nu} + \partial_{\mu}\partial_{\nu}g_{\beta\alpha}\right\},$$

where $\eta_{\mu\nu}$ is the metric of special relativity. **[5]**

(c) The energy-momentum tensor of a perfect fluid is $T^{\mu\nu} = (\rho + P/c^2)U^{\mu}U^{\nu} - Pg^{\mu\nu}$. Explain how this form can be justified by using the principle of manifest covariance.

Write down Einstein's field equations including a non-zero cosmological constant, Λ. Show that in the case of a non-relativistic fluid with negligible pressure, the Ricci scalar is $R = 4\Lambda + \kappa\rho c^2$, where $\kappa = 8\pi G/c^4$.

Using the result of part (b), obtain an approximation for R_{00} and hence show that the Newtonian limit of the field equations is

$$\nabla^2\Phi + \Lambda c^2 = 4\pi G\rho.$$

Explain how this permits a static universe, in which both ρ and Φ are constants. **[7]**

(d) The Kerr metric describes a black hole endowed with mass M and angular momentum J. In the equatorial plane ($\theta = \pi/2$) the line element is

$$c^2 d\tau^2 = c^2 dt^2 - (2\mu/r)(c\,dt - a\,d\phi)^2 - \frac{dr^2}{1 - 2\mu/r + a^2/r^2} - (r^2 + a^2)\,d\phi^2,$$

where $\mu = GM/c^2$ and $a = J/Mc$. An observer at large distance projects a particle radially towards the black hole. Show that the particle is subsequently observed to rotate around the hole, with

$$\frac{d\phi}{dt} = \frac{2ac\mu}{r^3 + a^2(r + 2\mu)}.$$

[6]