School of Physics & Astronomy

THE UNIVERSITY *of* EDINBURGH

General Relativity

PHYS11010 (SCQF Level 11)

Wednesday 3rd May, 2023 13:00 - 15:00
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **BOTH** questions

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous barcodes to *each* script book used.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof . K Rice
External Examiner: Prof. I McCarthy

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1.

(a) Explain the following terms and discuss the relation between them: weak equivalence principle; strong equivalence principle; local inertial frame; covariant equation; manifestly covariant equation. The equation $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$ is not manifestly covariant. By considering the covariant derivative,

$$\nabla_\nu V_\mu \equiv \partial_\nu V_\mu - \Gamma^\alpha_{\mu\nu} V_\alpha,$$

prove that the equation is in fact covariant. **[5]**

(b) Consider the line element

$$c^2 d\tau^2 = c^2 dt^2 - \frac{R^2}{r^2 + \alpha^2}\, dr^2 - R^2 d\theta^2 - (r^2 + \alpha^2) \sin^2\theta \, d\phi^2,$$

where $R^2 = r^2 + \alpha^2 \cos^2\theta$ and α is a constant. Find the geodesic equation for ϕ and show that it can be integrated to yield

$$\dot\phi \equiv \frac{d\phi}{d\tau} = \frac{J}{(r^2 + \alpha^2)\sin^2\theta},$$

where J is a constant. What is the geodesic equation for t? **[5]**

(c) Show that the geodesic equation for θ can be solved by $\theta = \pi/2$ and $\dot\theta = 0$. For this value of θ, show that the geodesic equation for r has the following integral form

$$\frac{r^2}{r^2 + \alpha^2}\, \dot r^2 + \frac{J^2}{r^2 + \alpha^2} = B^2$$

and that a solution is $r = \sqrt{D^2 + (v\tau)^2}$, where D and v are constants. Find B and J in terms of v, D and α. **[7]**

(d) Consider the coordinate transformation

$$\begin{aligned} x &= \sqrt{r^2 + \alpha^2} \sin\theta \cos\phi \\ y &= \sqrt{r^2 + \alpha^2} \sin\theta \sin\phi \\ z &= r\cos\theta, \end{aligned}$$

while t remains unchanged. The original metric takes the form

$$c^2 d\tau^2 = c^2 dt^2 - f^2 dx^2 - g^2 dy^2 - h^2 dz^2.$$

Find the functions f, g and h; what interpretation can now be given to the original metric? **[8]**

2.

(a) Justify the form of the energy-momentum tensor for a perfect fluid:

$$T^{\mu\nu} = (\rho + p/c^2)U^\mu U^\nu - p g^{\mu\nu}.$$

Write down the general law for the conservation of energy and momentum in terms of $T^{\mu\nu}$. Using $U_\mu U^\mu = c^2$ and noting that $U_\mu \nabla_\nu U^\mu = U^\mu \nabla_\nu U_\mu$, prove that if the pressure vanishes, elements of the fluid move on geodesics. **[7]**

(b) A one-dimensional model universe has a metric where the spatial part is periodic:

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) d\phi^2,$$

i.e. $\phi + 2\pi$ corresponds to the same point as ϕ. Show that there is a conserved quantity involving $d\phi/d\tau$. Using the invariant norm of the 4-velocity, obtain an equation for $dt/d\tau$ as a function of time. **[4]**

(c) A comoving observer in this cosmology ejects a particle with velocity v at time $t = t_0$ and scale factor $R(t_0) = R_0$. Explain why the initial value of $dt/d\tau$ is the Lorentz factor of the particle. Hence write an expression for $\phi(t)$ as an integral over a function of cosmological time, t. If the scale factor increases linearly with time, show that the velocity must exceed some minimum value if the particle is to be able to travel round the universe and return to its starting point:

$$\sinh^{-1}\left[\frac{\beta}{\sqrt{1-\beta^2}}\right] > 2\pi \,\frac{R_0}{ct_0},$$

where $\beta = v/c$ [hint: you may assume that $\int_1^\infty x^{-1}(1 + A^2x^2)^{-1/2} = \sinh^{-1}(1/A)$]. **[6]**

(d) Consider a spacetime where the line element takes the following form in cylindrical polars:

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 + f^2(r)\, d\phi^2 + dz^2.$$

Show that the only non-zero components of the affine connection are $\Gamma^r_{\phi\phi} = -ff'$ and $\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = f'/f$, where primes denote derivatives with respect to r. The Ricci tensor is

$$R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu},$$

and it may be assumed here that the only non-zero components are R_{rr} and $R_{\phi\phi}$. Hence show that

$$T^{\mu\nu} = \frac{c^4}{8\pi G}\,\frac{f''}{f}\,\mathrm{diag}(-1, 0, 0, 1).$$

[8]

Invigilators Information

Delivered papers 30 minutes prior to start time.

1. Course information

Course title:	General Relativity		
Course No:	PHYS11010		
Academic :	Prof. John Peacock	**Contact No:**	07946 273 597
Secretary:	Louise McCarte	**Contact No:**	0131 668 8403
School:	School of Physics & Astronomy	**Contact No:**	68-8261 / 51-7525

2. Exam diet / paper information

Date of exam:	Wednesday 3rd May, 2023	**Time of exam:**	13:00 - 15:00
Location of exam:	McEwan Hall		
No of exam papers:	55		

3. Invigilators Instructions

Unissued paper returned:	Yes	**Approved Calculators :**	Yes
Answer on exam paper:	No	**Open Book Exam:**	No
Answer on MCQ :	No	**Script book per answer:**	See rubric
Used exam papers returned:	No	**Dictionary allowed:**	No

4. Stationery Requirement

Stationery: 20 pages x 1.

5. Items to be handed out with exam papers

Calculators (from School):	No	**Barcodes (from School):**	Yes
Graph paper (from School:	No	**Bibles (from School):**	No
Formula Sheets (from School):	Yes		

6. Additional Information

- None

Notes: