School of Physics & Astronomy

THE UNIVERSITY *of* EDINBURGH

General Relativity

PHYS11010 (SCQF Level 11)

Monday 23rd May, 2022 13:00 – 15:00
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **BOTH** questions

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous barcodes to *each* script book used.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof P Best
External Examiner: Prof I McCarthy

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

1.

(a) If a metric is diagonal, show that the affine connection $\Gamma^{\alpha}_{\mu\nu}$ vanishes unless at least two of its indices are equal, and that the only non-zero terms are

$$\Gamma^{\alpha}_{\beta\beta} = -\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\beta\beta}}{\partial x^{\alpha}}; \quad \Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}}; \quad \Gamma^{\alpha}_{\alpha\alpha} = \frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}},$$

where in this instance repetition of indices does not imply summation, and α and β are distinct. **[5]**

(b) The line element on the surface of a sphere of radius a can be written as $d\ell^2 = dX^2 + dY^2 + dZ^2$ with the constraint $X^2 + Y^2 + Z^2 = a^2$. Show that this can be written as $d\ell^2 = dR^2\, a^2/(a^2 - R^2) + R^2\, d\phi^2$, giving expressions that show how Cartesian coordinates (X, Y, Z) are related to some new coordinates R and ϕ.

Assume that a hyperbolic surface of constant negative curvature will have a metric of the same form, but with $a^2 \to -a^2$. The Klein disc maps this infinite space onto a disc of unit radius, via the transformation $r = R/(a^2 + R^2)^{1/2}$. Show that the line element can then be written as

$$d\ell^2/a^2 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{1-r^2}\, d\phi^2.$$

[5]

(c) Carry out a coordinate transformation to Cartesian coordinates in the Klein disk (i.e. with $x^2 + y^2 = r^2$) and show that the metric changes from $g_{rr} = a^2/(1-r^2)^2$, $g_{\phi\phi} = a^2 r^2/(1-r^2)$ to

$$g_{xx} = a^2 \frac{1-y^2}{(1-r^2)^2}; \quad g_{yy} = a^2 \frac{1-x^2}{(1-r^2)^2}; \quad g_{xy} = a^2 \frac{1-xy}{(1-r^2)^2}.$$

[5]

(d) Using the Lagrangian formalism, or otherwise, compute the equations obeyed by geodesics on the Klein disc in (r, ϕ) space, and prove that they are straight lines (hint: for a suitable origin of ϕ, a straight line in 2D polars is $r \propto 1/\cos\phi$). Given a geodesic passing through a point, A, demonstrate graphically that there are many geodesics passing through a different point, B, which are all parallel to the first (i.e. which do not meet it even when extended without limit). **[10]**

2. (a) Explain why the 4-acceleration as defined in Special Relativity, $A^\mu = dU^\mu/d\tau$ is not a general 4-vector (where U^μ is 4-velocity and τ is proper time). An alternative form using the covariant derivative, ∇_μ, does however permit a covariant generalization of the equation of motion for a freely-falling particle:

$$A^\mu = U^\alpha \nabla_\alpha U^\mu = 0; \quad \nabla_\alpha U^\mu \equiv \partial/\partial x^\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\beta.$$

Show that this equation reduces to the usual geodesic equation for $d^2x^\mu/d\tau^2$. **[5]**

(b) Covariant differentiation of a tensor commutes with contraction. Prove this result for the case of a 2nd-rank tensor $T^{\mu\nu}$, i.e. consider $\nabla_\alpha T^\mu{}_\nu$, set $\mu = \nu$ after differentiation, and prove that this is the same as $\nabla_\alpha (T^\mu{}_\mu)$. **[5]**

(c) Consider a stationary observer at radius r in the Schwarzschild spacetime and show that their 4-acceleration is

$$A^\mu = (0, GM/r^2, 0, 0).$$

(hint: compute Γ^r_{tt} from the equation of motion for r. Results from Q1 may be relevant here). Now compute the proper acceleration $a = (A^\mu A_\mu)^{1/2}$. Show that this agrees with the Newtonian expectation for $r \gg 2GM/c^2$ and that stationary observers cannot exist for $r < 2GM/c^2$. **[5]**

(d) A massive particle is projected towards a Schwarzschild black hole of mass M, with impact parameter b and non-relativistic velocity v at large distance from the hole. Show that the particle will be captured by the black hole if b is below a critical value, b_c, and that the capture cross-section is

$$\sigma \equiv \pi b_c^2 = 16\pi (GM/c^2)^2/(v/c)^2$$

(hint: if the particle is not captured, there must be a potential barrier at some minimum radius. How high must this be in the extreme non-relativistic limit where the kinetic energy at infinity goes to zero?).

What is the corresponding capture cross-section for light? **[10]**