College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, PHYS11010, PHY-5-GenRel

26 April 2021
13.00p.m. - 15.00p.m.

Chairman of Examiners
Prof P. Best

External Examiner
Prof L. Miller

Answer **BOTH** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

PRINTED: JANUARY 11, 2022

1.

(a) Write brief notes on the concept of general covariance, explaining how the equivalence principle allows us to use special relativity as a basis for fully relativistic laws of physics. **[4]**

(b) Consider the line element of 2D Euclidean space in polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Using the Lagrangian approach, or otherwise, obtain the geodesic equation for this space and show that this implies the following equation:

$$(dr/d\theta)^2 + r^2 = kr^4,$$

where k is a constant. Write down the equation of a straight line in this space, and show that it satisfies the above relation. **[5]**

(c) An observer in special relativity undergoes the $x(t)$ trajectory

$$\begin{aligned} x &= \alpha^{-1}\cosh(\alpha c\tau) - \alpha^{-1} \\ ct &= \alpha^{-1}\sinh(\alpha c\tau), \end{aligned}$$

where τ is proper time and α is a constant. Using the invariant norm of the 4-acceleration, $A^\mu A_\mu$, show that the observer perceives a constant acceleration. The above trajectory can be extended to a set of accelerated observers by defining spatial and time coordinates (x', t') in the accelerated frame:

$$\begin{aligned} x &= (x' + \alpha^{-1})\cosh(\alpha ct') - \alpha^{-1} \\ ct &= (x' + \alpha^{-1})\sinh(\alpha ct'), \end{aligned}$$

Show that these transformations define the line element of Rindler Space:

$$c^2 d\tau^2 = (1 + \alpha x')^2 c^2 dt'^2 - dx'^2.$$

Discuss the form of this metric from the point of view of gravitational time dilation. For this 2D metric, compute the non-zero affine connections and hence show that the Riemann tensor vanishes, as expected. **[6]**

(d) Two test particles are released from rest at the same coordinate time t around a Schwarzschild black hole of mass M. They have the same θ and ϕ and radii r and $r + \Delta r$, both at $r > 2GM/c^2$. Each particle has an initial 4-velocity $U^\alpha = (U^0, 0, 0, 0)$; use the invariant $U^\mu U_\mu = c^2$ to compute U^0.

Use the equation of geodesic deviation to express the dependence of the radial separation on proper time. What component of the Riemann curvature tensor appears here? Because the metric is independent of time, the relevant components of the Affine connection are

$$\begin{aligned} \Gamma^1{}_{00} &= -\frac{1}{2}g^{11}\partial_1 g_{00} = (1 - 2\mu/r)\mu/r^2 \\ \Gamma^1{}_{11} &= \frac{1}{2}g^{11}\partial_1 g_{11} = (1 - 2\mu/r)^{-1}\mu/r^2 \\ \Gamma^0{}_{01} = \Gamma^0{}_{10} &= \frac{1}{2}g^{00}\partial_1 g_{00} = -(1 - 2\mu/r)^{-1}\mu/r^2, \end{aligned}$$

where $\mu \equiv GM/c^2$. Hence compute the necessary component of the Riemann tensor and show that the expression for the effective radial tidal force has the same form as the Newtonian formula for the tide at radius r around a point mass M.

Suppose that the sun (mass $m_\odot \simeq 2 \times 10^{30}\,\mathrm{kg}$ and radius $R_\odot \simeq 7 \times 10^8\,\mathrm{m}$) fell radially towards a Schwarzschild black hole of mass M. Give the order of magnitude of M for which the sun will just survive being tidally disrupted as it passes through the event horizon (You will need $G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$). What mass would allow a human to pass through the horizon unscathed? **[10]**

2.

(a) Briefly discuss the concept of parallel transport of a vector, explaining how it can be used to measure the curvature of a manifold, and how it can be used to define a covariant derivative. **[4]**

(b) A gauge transformation is an infinitesimal coordinate transformation of the form

$$x'^\mu = x^\mu + \xi^\mu,$$

where ξ^μ is an arbitrary function of spacetime coordinates. Working to first order in the small quantity ξ^μ, derive the effect of a gauge transformation on a scalar, $\phi(x^\mu)$, and on a vector, $V^\alpha(x^\mu)$. This is different to the question we usually ask, which is how does the field at a given point change when we alter the coordinates: here we are working at fixed coordinate value, so the point under consideration changes when we make the gauge transformation. So ask how $\phi'(x'^\mu = a^\mu)$ differs from $\phi(x^\mu = a^\mu)$, where a^μ is some fixed reference value. **[4]**

(c) Show that the geodesic equation of motion for a nonrelativistic particle with $U^\mu \simeq (c, \mathbf{v})$ reduces to the following equation to first order in the particle velocity:

$$\frac{d^2 x^i}{dt^2} = -c^2 \Gamma^i_{00} - 2c\Gamma^i_{0j} v^j + c\Gamma^0_{00} v^i.$$

(note derivatives with respect to time, not proper time: change variable in $d^2x^\mu/d\tau^2$). If the gravitational field is weak and static, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \ll 1$, show that this expression contains a gravitational analogue of magnetic force:

$$\frac{d^2 \mathbf{x}}{dt^2} = -\boldsymbol{\nabla}\Phi + \mathbf{v} \wedge (\boldsymbol{\nabla} \wedge \mathbf{A}),$$

where $\Phi = (c^2/2)h_{00}$ and $A_i = -ch_{0i}$ (write the cross product in terms of components: $[\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})]_i = A_j B_i C_j - A_j B_j C_i$). **[5]**

(d) An alternative relativistic theory of gravity was proposed by Nordstrøm in 1913, in which there is a single field equation,

$$R = \kappa\, g_{\mu\nu} T^{\mu\nu},$$

where R is the Ricci scalar. Nordstrøm's theory also assumed the metric to be 'conformally flat':

$$g^{\mu\nu} = \exp(2\phi)\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the metric of special relativity and ϕ is an arbitrary function of spacetime coordinates. Show that, with a suitable interpretation of ϕ, this theory gives the correct weak-field equation of motion for nonrelativistic particles. Compute the Newtonian limit of the field equation and hence deduce the constant κ (use the same approach as for the Newtonian limit of Einstein's equations). But now show that the theory is incompatible with observations, as it predicts that light rays undergo no deflection in a gravitational field. **[5]**

(e) E.A. Milne (no relation to Winnie the Pooh) proposed a kinematical cosmology, in which fundamental observers are ejected from an origin at $t = 0$ with a range of velocities. But the matter density is negligible, so the background spacetime is Minkowski. Show that this model automatically has a linear distance-redshift relation; what is the Hubble parameter at time t? Show that the cosmological time t' is related to time in the Minkowski background via time dilation:

$$t' = t/\gamma = t\,\sqrt{1 - r^2/c^2t^2}.$$

Hence show that r can be eliminated in terms of cosmological time and Lorentz factor:

$$r = ct'\sqrt{\gamma^2 - 1}.$$

If we now define the velocity variable ω,

$$v/c = \tanh\omega \quad \Rightarrow \quad \gamma = \cosh\omega,$$

show that the metric can therefore be written in the $k = -1$ Robertson–Walker form:

$$d\tau^2 = dt'^2 - t'^2\left(d\omega^2 + \sinh^2\!\omega\; d\psi^2\right).$$

where $d\psi$ is a element of angle on the sky. Integrate a radial null geodesic to show that the redshift of a photon emitted at 'radius' ω_e and received at the origin ($\omega = 0$) is

$$1 + z = \exp(\omega_e).$$

Is this what we would expect based on special relativity in the initial Minkowski frame? **[7]**