College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, PHYS11010, PHY-5-GenRel

???, 2020
9.30a.m. - 11.30a.m.

Chairman of Examiners
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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

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1. (a) Explain what is meant by the term inertial force in Newtonian mechanics, and show that such a force can be created by a coordinate transformation to the frame of an accelerating observer: $(t', \mathbf{x}') = (t, \mathbf{x} - \mathbf{a}t^2/2)$. Define inertial and gravitational mass, and discuss how their observed equality led Einstein to postulate the Weak and Strong Equivalence Principles. **[4]**

(b) Write down the equation of motion in Special Relativity for a free particle in a local inertial frame. Using the Weak Equivalence Principle, show that this becomes the Geodesic Equation in a general frame of reference:

$$\frac{dU^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu}\, U^\mu U^\nu = 0.$$

Define all terms carefully. **[4]**

(c) Show that this approach also requires spacetime to have a metric tensor, $g_{\mu\nu}$, and that the additional coefficient in the Geodesic Equation is

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\eta} (\partial_\mu g_{\nu\eta} + \partial_\nu g_{\mu\eta} - \partial_\eta g_{\mu\nu}),$$

where $\partial_\mu = \partial/\partial x^\mu$. **[4]**

(d) Explain the concept of parallel transport and how it can be used to define the covariant derivative of a vector in General Relativity: $\nabla_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu{}_{\alpha\nu} V^\alpha$. Hence show that the Geodesic Equation can be written as $U^\nu \nabla_\nu U^\mu = 0$. **[5]**

(e) By considering suitable products of 4-vectors, derive an expression for the covariant derivative of a covector, $\nabla_\nu V_\mu$. What is the covariant derivative of the tensor $T^\alpha{}_{\mu\nu}$? **[3]**

(f) Prove the following relation for the commutator of covariant derivatives:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, V^\mu = R^\mu{}_{\nu\alpha\beta}\, V^\nu,$$

where the Riemann Tensor is

$$R^\alpha{}_{\sigma\rho\beta} \equiv \partial_\rho \Gamma^\alpha{}_{\beta\sigma} - \partial_\beta \Gamma^\alpha{}_{\rho\sigma} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\nu{}_{\sigma\beta} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\nu{}_{\sigma\rho}$$

[5]

2. (a) By considering the rest frame of a freely-falling massive particle, or otherwise, show that it follows a trajectory of maximum proper time. Hence argue that it should be possible to obtain equations of motion using a Lagrangian $L = (g_{\mu\nu}U^\mu U^\nu)^{1/2}$, where $g_{\mu\nu}$ is the metric tensor and U^μ is the 4-velocity. Explain why it is possible to use the Lagrangian formalism with $L \to L^2$, and say how this approach can be adapted to the case of a massless particle. **[5]**

(b) For a static and spherically symmetric spacetime, explain why it is possible to write the line element in terms of two unknown functions $A(r)$ and $B(r)$:

$$c^2 d\tau^2 = A(r)c^2 dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Write down the Euler-Lagrange equations of motion and hence obtain the following components of the Affine Connection: $\Gamma^t{}_{rr}$, $\Gamma^t{}_{tr}$, $\Gamma^r{}_{tt}$, $\Gamma^r{}_{tr}$. **[5]**

(c) The resulting non-zero components of the Ricci tensor in this case are

$$\begin{aligned}
R^t{}_t &= -\frac{A''}{2AB} + \frac{A'B'}{4AB^2} + \frac{A'^2}{4A^2B} - \frac{A'}{rAB} \\
R^r{}_r &= -\frac{A''}{2AB} + \frac{A'B'}{4AB^2} + \frac{A'^2}{4A^2B} + \frac{B'}{rB^2} \\
R^\theta{}_\theta &= R^\phi{}_\phi = -\frac{A'}{2rAB} + \frac{B'}{2rB^2} + \frac{B-1}{r^2B}.
\end{aligned}$$

Assume that we seek a solution for the field of a point mass surrounded by vacuum, but with a non-zero cosmological constant. Prove that the normal vacuum field equation $G^{\mu\nu} + \Lambda g^{\mu\nu} = 0$ can be recast as $R^{\mu\nu} = \Lambda g^{\mu\nu}$. Use appropriate combinations of the above elements of the Ricci tensor to prove that we can take

$$A = 1/B = 1 - \frac{2GM}{c^2 r} - \Lambda\frac{r^2}{3}.$$

[5]

(d) For this Schwarzschild–de Sitter metric, obtain the Euler–Lagrange equations for a particle in an equatorial orbit, and show that they imply the conservation equations

$$A\dot{t} = k \qquad \text{and} \qquad r^2\dot{\phi} = h,$$

where k and h are constants. Using these relations, give an expression for the effective potential that governs the radial motion of a massive particle in orbit in this spacetime:

$$\frac{\dot{r}^2}{2} + \Phi_{\rm eff} = \frac{k^2c^2}{2}.$$

[5]

(e) Now consider the motion of a photon in this spacetime and show that the effective potential is modified. Show that there is only one circular orbit for a photon, and say how its radius depends on Λ. Is this orbit stable or unstable? **[5]**

3. (a) Explain the relation between the Principles of General Covariance and Manifest Covariance. Illustrate your answer by showing how the behaviour of a covector, U_μ, under a coordinate transformation can be derived from the transformation of a 4-vector:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu .$$

[5]

(b) Consider the separation, Δx^λ, of points on two nearby geodesics. Assuming without proof the geodesic equation of motion, and working in a local inertial frame, derive the equation of geodesic deviation in the form

$$\frac{D^2 \Delta x^\lambda}{d\tau^2} + \left(R^\lambda{}_{\alpha\nu\beta}\dot{x}^\alpha \dot{x}^\beta\right) \ \Delta x^\nu = 0,$$

where $D/d\tau$ is a covariant derivative and the Riemann tensor is $R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha{}_{\nu\mu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\beta\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\eta}\Gamma^\eta{}_{\mu\beta}$. **[10]**

(c) The relativistic energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p/c^2)U^\mu U^\nu - p g^{\mu\nu},$$

where ρ is the rest-frame density, p is the pressure, and U^μ is the 4-velocity of the fluid flow. Explain why this tensor obeys the following conservation equation:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu{}_{\alpha\nu} T^{\alpha\mu} = 0.$$

[4]

(d) Consider the energy-momentum tensor for a perfect fluid in Special Relativity, and show that the conservation law from part (c) yields the following equations of relativistic hydrodynamics:

$$\frac{d}{dt}\mathbf{v} = -\frac{1}{\gamma^2(\rho + p/c^2)} \left(\boldsymbol{\nabla} p + \dot{p}\mathbf{v}/c^2\right);$$

$$\frac{d}{dt}\left[\gamma^2(\rho + p/c^2)\right] = \dot{p}/c^2 - \gamma^2(\rho + p/c^2)\boldsymbol{\nabla}\cdot\mathbf{v},$$

where $\dot{p} \equiv \partial p/\partial t$, γ is the Lorentz factor of the fluid flow, and $d/dt \equiv \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$.

[6]