College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, PHYS11010, PHY-5-GenRel

???, 2019
9.30a.m. - 11.30a.m.

Chairman of Examiners
Prof P. Best

External Examiner
Prof L. Miller

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

PRINTED: FEBRUARY 25, 2020

1. (a) Explain the *Weak Equivalence Principle.* Why does the equivalence of inertial and gravitational mass imply the curvature of spacetime? **[5]**

(b) Starting from the Weak Equivalence Principle derive the Geodesic Equation for a free particle in a gravitational field,

$$\frac{du^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu}\, u^\mu u^\nu = 0,$$

defining each term. **[5]**

(c) Consider a particle moving in a weak gravitational field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$ is static. Show that in the slow-motion limit ($|u^i| \ll u^0$) the Geodesic Equation reduces to the Newtonian equation of motion,

$$\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla}\Phi,$$

where Φ is the Newtonian potential. **[7]**

(d) In a weak gravitational field with static potential, Φ, the spacetime line element is

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j.$$

Show that a massless particle with 4-velocity $u^\mu = c(1, \hat{\boldsymbol{n}})$, where $\hat{\boldsymbol{n}}$ is a unit vector in the direction of propagation, obeys the relativistic equation of motion

$$\ddot{\boldsymbol{x}} = 2\,\hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}} \times \boldsymbol{\nabla}\Phi.$$

You may assume to lowest order the time parameter is the coordinate time, t, and $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{\nabla} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{\nabla}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{\nabla}$. **[5]**

(e) Compare and contrast the equations of motion for non-relativistic and massless particles. **[3]**

2. The spacetime line element around a static black hole with mass M is

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi_s}{c^2}\right) c^2 dt^2 - \left(1 + \frac{2\Phi_s}{c^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $\Phi_s = -GM/r$.

(a) For a particle in an equatorial orbit show that the Euler-Lagrange equations imply the conservation equations,

$$\left(1 + \frac{2\Phi_s}{c^2}\right)\dot{t} = k \qquad \text{and} \qquad r^2\dot{\varphi} = h,$$

where k and h are constants. Explain what the conserved quantities represent. **[5]**

(c) Show that the radial motion of a massive particle in orbit in this spacetime obeys the energy equation

$$\frac{1}{2}\dot{r}^2 + \Phi_s + \left(1 + \frac{2\Phi_s}{c^2}\right)\frac{h^2}{2r^2} = \frac{1}{2}c^2\left(k^2 - 1\right).$$

Define the effective potential for orbits in this spacetime, explaining each term.

Using the effective potential illustrate the possible orbits of a particle moving in the equatorial plane. **[10]**

(d) Consider an observer who is stationary just outside the Schwarzschild radius, r_s. A massive particle freely falling from an infinite radius passes this observer. What speed does the stationary observer measure the particle to be travelling at, according to the observer's proper time and distance? **[5]**

(e) The Schwarzschild spacetime can be maximally extended by the Kruskal-Szekeres coordinates,

$$T = \left|1 - \frac{r}{r_s}\right|^{1/2} e^{r/2r_s}\sinh\frac{ct}{2r_s}, \qquad X = \left|1 - \frac{r}{r_s}\right|^{1/2} e^{r/2r_s}\cosh\frac{ct}{2r_s},$$

when $r > r_s$ and

$$T = \left|1 - \frac{r}{r_s}\right|^{1/2} e^{r/2r_s}\cosh\frac{ct}{2r_s}, \qquad X = \left|1 - \frac{r}{r_s}\right|^{1/2} e^{r/2r_s}\sinh\frac{ct}{2r_s},$$

when $r < r_s$. Sketch this spacetime in the T, X plane, indicating the relevant features.

Describe what an observer falling into the black hole will see in both their forward and backward directions for $r > r_s$ and $0 < r < r_s$. **[5]**

3. (a) Explain the *Principle of General Covariance.* **[5]**

(b) Show how the spacetime derivative of a 4-vector, $\partial_\lambda A^\mu$, and a tensor, $\partial_\lambda B^{\mu\nu}$, can be written in covariant form. Express this in terms of the affine connections. **[5]**

(c) The Robertson-Walker line element for a spatially flat, expanding, homogeneous and isotropic universe is

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t) dx^2,$$

where $dx^2 = \delta_{ij}\, dx^i dx^j$. Using the Euler-Lagrange equations, or otherwise, find the Geodesic Equations for t and hence the affine connections, ${\Gamma^0}_{\nu\mu}$. **[5]**

(d) Show that the covariance divergence of a 4-vector can be written

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g} A^\mu\right).$$

You may use the matrix identity $\ln \mathrm{Det}\, \boldsymbol{M} = \mathrm{Tr}\, \ln \boldsymbol{M}$, for an arbitrary matrix $\boldsymbol{M}$. The determinant of the spacetime metric is $\mathrm{Det}\, g_{\mu\nu} = -g$, and $\mathrm{Tr}\, B_{\mu\nu} = B_{\mu\nu}\delta^\mu_\nu$ is the trace (contraction) of a tensor.

Hence show that the covariant divergence of $B^{\mu\nu}$ can be written

$$\nabla_\mu B^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} B^{\mu\nu}\right) + {\Gamma^\nu}_{\mu\rho} B^{\mu\rho}.$$

[5]

(e) The relativistic stress-energy tensor for a perfect fluid is

$$T^{\mu\nu} = \rho u^\mu u^\nu + \frac{P}{c^2}(u^\mu u^\nu - c^2 g^{\mu\nu}),$$

where ρ is the density and P is the pressure. Let the fluid be uniform and expanding so that it is at rest with respect to the comoving x-coordinates in the Robertson-Walker spacetime, $u^\mu = (c, \mathbf{0})$. Show that $\nabla_\mu T^{\mu 0} = 0$ leads to the conservation of energy equation in an expanding universe,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0,$$

where there a dot denotes a derivative with respect to coordinate time. **[5]**