College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, PHYS11010, PHY-5-GenRel

???, 2018
9.30a.m. - 11.30a.m.

Chairman of Examiners
Prof J. S. Dunlop

External Examiner
Prof L. Miller

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

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DRAFT

1. (a) Define the terms *Weak* and *Strong Equivalence Principles* in GR. Why does the Equivalence Principle imply gravity is not a force like electromagnetism? **[5]**

(b) Starting from the Weak Equivalence Principle show that the equation of motion for a particle moving freely in a gravitational field obeys the Geodesic equation,

$$\frac{du^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu}\, u^\mu u^\nu = 0,$$

where $u^\lambda = dx^\lambda/d\tau$ is the 4-velocity vector and $\Gamma^\lambda{}_{\mu\nu}$ is the affine connection. **[5]**

(c) Write down how the metric tensor, $g_{\mu\nu}$, transforms between the local inertial frame and a general coordinate system. Hence show that

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\eta}(\partial_\mu g_{\mu\eta} + \partial_\nu g_{\mu\eta} - \partial_\eta g_{\mu\nu}).$$

where $\partial_\mu = \partial/\partial x^\mu$. **[5]**

(d) Show that in the weak-field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, slow-motion limit, keeping terms of order $u^0 u^i$, the Geodesic equation reduces to the Lorentz-like equation,

$$\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla}\Phi + \boldsymbol{v} \times \boldsymbol{B}_g,$$

where $\boldsymbol{B}_g = \boldsymbol{\nabla} \times \boldsymbol{A}_g$, and the gravitational vector potential is $A_{g,i} = h_{0i}c$. **[5]**

(e) The Earth has angular momentum $J_E = 2MR^2\Omega/5$, where R is its radius, M its mass, and Ω its rotational frequency. The gravo-magnetic field an equatorial distance r from the Earth has an amplitude of $|\boldsymbol{B}_g| = 2GJ_E/c^2r^3$. Estimate the relative gravo-magnetic acceleration compared to the Newtonian gravitational field for a satellite in a geosynchronous equatorial orbit. **[5]**

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2. The spacetime in the equatorial plane ($\theta = \pi/2$) around a rotating black hole with mass M, and angular momentum J, is described by the Kerr line element,

$$c^2 d\tau^2 = c^2 dt^2 - \frac{r_S}{r}\,(c dt - r_J d\varphi)^2 - \frac{dr^2}{1 + r_J^2/r^2 - r_S/r} - (r^2 + r_J^2)\, d\varphi^2,$$

where $r_S = 2GM/c^2$ is the Schwarzschild radius, and $r_J = J/Mc$ is a length scale associated with the rotation of the black hole.

(a) Show that as $M \to 0$, for fixed r_J, the Kerr spacetime reduces to Minkowski spacetime. **[2]**

(b) When $J = 0$, show the Euler-Lagrange equations imply the conservation equations $(1 - r_S/r)\dot{t} = k$ and $r^2\dot{\varphi} = h$ for a particle in an equatorial orbit. **[5]**

(c) Show that the radial motion of a massive particle in orbit in this spacetime obeys the energy equation

$$\dot{r}^2 - \frac{r_S c^2}{r} + \frac{h^2}{r^2} - \frac{r_S h^2}{r^3} = c^2(k^2 - 1).$$

Define the effective potential for orbits in this spacetime, explaining each term.

Illustrate the possible orbits using the effective potential and the motion of the particle in the equatorial plane. **[10]**

(d) Consider a particle on a circular orbit in the Kerr spacetime with angular speed $\Omega = d\varphi/dt$ at fixed $r = r_S$. Using the line element show that to maintain a timelike trajectory,

$$0 < \Omega < \frac{2\,c\,r_J}{2r_J^2 + r_S^2}.$$

[3]

(e) A real singularity exists for the Kerr spacetime at $r = 0$. Describe the structure of this singularity compared to the singularity in a Schwarzschild metric.

The radial distance in the Kerr spacetime can be continued to $r < 0$. Consider a particle on an orbit just inside $r = 0$ with $dt = dr = 0$. What does it experience?

[5]

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3. (a) Explain what the *Principle of General Covariance* is. **[5]**

(b) The Robertson-Walker line element for a spatially flat, expanding, homogeneous and isotropic universe is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) dx^2,$$

where $dx^2 = \delta_{ij}\, dx^i dx^j$. Using the Euler-Lagrange equations, or otherwise, find the Geodesic equations for t and x^i, and hence the affine connections, $\Gamma^\alpha{}_{\nu\mu}$. **[5]**

(c) The Einstein equations for empty space with a cosmological constant, Λ, are

$$R_{\mu\nu} = -\Lambda g_{\mu\nu},$$

where $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha{}_{\nu\mu} - \partial_\nu \Gamma^\alpha{}_{\alpha\mu} + \Gamma^\alpha{}_{\alpha\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\eta}\Gamma^\eta{}_{\mu\alpha}$ is the Ricci tensor. Derive the $t-t$ Einstein equation and show that the scale factor is

$$R(t) = R_0 e^{\sqrt{\Lambda/3}\, c(t-t_0)},$$

where $R = R_0$ at time t_0. **[10]**

(d) A photon is emitted from a source at coordinate distance x at time t_i and is received by an observer at time t in this spacetime. Show this distance is

$$x = \frac{\sqrt{3}}{\sqrt{\Lambda} R_0}\left(e^{-\sqrt{\Lambda/3}\, c(t_i-t_0)} - e^{-\sqrt{\Lambda/3}\, c(t-t_0)}\right).$$

Show that a photon emitted by the observer at time t_0 asymptotically approaches a maximum coordinate distance but never reaches it. **[5]**

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