College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2017
9.30a.m. - 11.30a.m.

Chairman of Examiners
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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

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1. (a) Explain what the *Weak* and *Strong Equivalence Principles* of General Relativity are. Why does the observed equivalence of inertial and gravitational mass imply the curvature of spacetime? **[5]**

(b) Use the Weak Equivalence Principle to show that the equation of motion for a particle moving freely in a gravitational field obeys the Geodesic Equation,

$$\frac{du^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu}\, u^\mu u^\nu = 0,$$

where $u^\lambda = dx^\lambda/d\tau$ is the 4-velocity vector and $\Gamma^\lambda{}_{\mu\nu}$ is the affine connection. **[5]**

(c) By considering the derivatives of the metric, or otherwise, derive the relationship between the affine connection and the metric. **[5]**

(d) What are vectors and co-vectors and how do they transform under a general coordinate change? What is the relationship between vectors and co-vectors?

Show that the Geodesic Equation leads to the co-vector 4-acceleration equation,

$$\frac{du_\lambda}{d\tau} = \frac{1}{2}\,\partial_\lambda g_{\mu\nu}\, u^\mu u^\nu.$$

The 4-momentum co-vector for a particle of mass m is $p_\lambda = m u_\lambda$. What symmetry must the metric exhibit for this to be conserved? What does this imply about the symmetries of the Lagrangian-squared? **[7]**

(e) Consider the Robertson-Walker spacetime for a spatially flat universe,

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)(dx^2 + dy^2 + dz^2),$$

where $R(t)$ is a cosmic scale factor and (x, y, z) are co-moving spatial coordinates. What are the conserved 4-momentum co-vectors for this spacetime? Show the corresponding 4-momentum vector decays and suggest a reason for this. **[3]**

2. (a) The Schwarzschild spacetime outside a static black hole with mass M is described by the line element

$$c^2 d\tau^2 = (1-\beta)c^2 dt^2 - \frac{dr^2}{1-\beta} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

with $\beta = r_S/r$, where $r_S = 2GM/c^2$ is the Schwarzschild radius. Using the Euler-Lagrange equations, or otherwise, derive the conservation equations $(1-\beta)\dot{t} = k$ and $r^2\dot{\varphi} = h$ for a particle moving in an equatorial orbit ($\theta = \pi/2$). **[5]**

(b) Show that the radial motion of a massive particle in orbit in this spacetime obeys the energy equation

$$\dot{r}^2 - \frac{r_S c^2}{r} + \frac{h^2}{r^2} - \frac{r_S h^2}{r^3} = c^2(k^2 - 1).$$

Define the effective potential for orbits in this spacetime, explaining what each term represents. Illustrate the possible orbits using the effective potential and the motion of the particle in the equatorial plane. **[10]**

(c) Assume a particle is in a stable, circular orbit in the Schwarzschild spacetime. What is the radius of this orbit? Show that Kepler's third law for this particle,

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3},$$

holds in General Relativity. [Hint: use $d\Phi/dr = 0$ to find h, and then the radial energy equation to find k.] **[5]**

(d) A massive particle falls radially into the black hole. Assuming the particle is at rest as $r \to \infty$, find the particle's trajectory, $r(\tau)$, in terms of its proper time, τ. What happens to the observed particle velocity as the particle reaches $r = r_S$? **[3]**

(e) The Schwarzschild line element is manifestly invariant under time reversal, $t \to -t$. However, an infalling particle crossing $r = r_S$ cannot re-emerge. Suggest how to resolve this apparent paradox. **[2]**

3. (a) Define the *Principle of General Covariance* and *tensors* and explain why tensor equations are of importance in General Relativity. **[5]**

(b) Explain why the Riemann tensor, $R^{\alpha}{}_{\mu\beta\nu}$, is the Generally Covariant extension of the Newtonian tidal field. Hence, by analogy with the Laplace equation, $\nabla^2\Phi = 0$, show that the Einstein Field Equations in empty space are given by $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor.

Explain briefly why the Einstein equations,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

are the relativistic generalisation of Newtonian gravity. **[10]**

(c) Einstein introduced his Cosmological Constant, Λ, a 100 years ago in February 1917. Why can a Cosmological Constant be added to the Einstein Equations?

Show that in a universe with a Cosmological Constant and no matter or radiation, so that $G_{\mu\nu} - \Lambda g_{\mu\nu} = 0$, the Einstein equations can be written

$$R_{\mu\nu} = \Lambda g_{\mu\nu}.$$ **[5]**

(d) The cosmological Lemaitre models include a Cosmological Constant. The Friedmann equation for these models can be written

$$\left(\frac{dR}{dt}\right)^2 = H_0^2(\Omega_m R^{-1} + \Omega_\Lambda R^2 + \Omega_K),$$

where H_0 is the current value of the Hubble parameter, $\Omega_m = 8\pi G\rho_{m,0}/(3H_0^2)$ is the current matter density parameter, $\Omega_\Lambda = \Lambda/(3H_0^2)$, and $\Omega_K = -kc^2/H_0^2$ is the curvature parameter, while $R(t)$ is the cosmological scale factor. Interpreting this as an energy equation, sketch the effective potential and expansion histories for this model, indicating when different terms dominate. Note Λ and k can be positive or negative.

Show that the transition from matter to Λ-domination happens at $R_* = (\Omega_m/2\Omega_\Lambda)^{1/3}$. Hence show these models can enter a "coasting" phase with no expansion when

$$\Omega_K = -3\left(\frac{\Omega_m^2\Omega_\Lambda}{4}\right)^{1/3}.$$

Explain, using the effective potential, how this solution arises. **[5]**