College of Science and Engineering
School of Physics

# General Relativity

**SCQF Level 11, U01429, PHY-5-GenRel**

**???, 2016**
**9.30a.m. - 11.30a.m.**

**Chairman of Examiners**
Prof J. S. Dunlop

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Answer **TWO** questions

**The bracketed numbers give an indication of the value assigned to each portion of a question.**

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

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**1.** (a) Define the *Weak* and *Strong Equivalence Principles*. Explain why the observed equivalence of inertial and gravitational mass implies the curvature of spacetime. **[5]**

(b) Starting from the Weak Equivalence Principle show that the equation of motion for a particle moving freely in an arbitrary gravitational field obeys the geodesic equation,

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the affine connection. **[7]**

(c) By considering two nearby points in a gravitational field separated by $\Delta x^\lambda$, use the geodesic equation to derive the geodesic deviation equation,

$$\frac{D^2\Delta x^\lambda}{d\tau^2} + \left(R^\lambda{}_{\alpha\nu\beta}\dot{x}^\alpha\dot{x}^\beta\right)\Delta x^\nu = 0,$$

where $R^\alpha{}_{\mu\beta\nu} = \partial_\beta\Gamma^\alpha{}_{\nu\mu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\beta\eta}\Gamma^\eta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\eta}\Gamma^\eta{}_{\mu\beta}$ is the Riemann tensor. How does this result connect gravitation to spacetime curvature? **[8]**

(d) The stress-energy tensor for a pressureless fluid, $T^{\mu\nu} = \rho u^\mu u^\nu$, is a covariantly conserved quantity,

$$\nabla_\mu T^{\mu\nu} = 0.$$

By contracting this with 4-velocity covector, $u_\nu$, show that the mass-density current, $j^\mu = \rho u^\mu$, is also conserved in a gravitational field [You may use that the covariant derivative of the expression $u^\nu u_\nu = c^2$ implies $u_\nu\nabla_\mu u^\nu = 0$].

Hence, show that covariant conservation of the stress-energy tensor implies the motion of a pressureless fluid in a gravitational field obeys the geodesic equation.

**[5]**

**2.** (a) The spacetime line element outside a static spherically symmetric mass distribution with mass $M$ is

$$c^2 d\tau^2 = (1-\beta)c^2 dt^2 - \frac{dr^2}{1-\beta} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $\beta = r_S/r$ and $r_S = 2GM/c^2$ is the Schwarzschild radius. Using the Euler-Lagrange equations derive the conservation equations $(1-\beta)\dot{t} = k$ and $r^2\dot{\varphi} = h$ for a particle moving in an equatorial orbit ($\theta = \pi/2$) around this object. **[5]**

(b) Show that the radial motion of a massive particle in this spacetime is governed by the energy equation

$$\dot{r}^2 - \frac{r_s c^2}{r} + \frac{h^2}{r^2} - \frac{r_s h^2}{r^3} = c^2(k^2-1).$$

Define the effective potential for orbits in this spacetime. **[8]**

(c) Use the effective potential to find the radius of stable and unstable orbits in the Schwarzschild spacetime.

Explain what the last stable orbit is. Show that the angular momentum of a particle on the last stable orbit is $h = \sqrt{3}r_s c$, and hence that its radius is $r = 3r_s$.

Use the energy equation to show that $k = 2\sqrt{2}/3$ for this particle. **[7]**

(d) The energy of a particle in a gravitational field is $E = p_0 c$, where $p_0 = m_0 g_{00} u^0$ and $m_0$ is its rest-mass. Using the conservation of particle energy show that in the gravitational field of a black hole $E = k m_0 c^2$.

Black hole accretion occurs when the angular momentum of free particles is low enough that there is no stable circular orbit and particles infall. Assuming that all of the energy of the particle is released when it falls in, argue that the efficiency of energy release from black hole accretion is $\epsilon_{\rm acc} \equiv \Delta E/E = 1 - k$.

Estimate the magnitude of $\epsilon_{\rm acc}$. How does this compare with the 0.7% efficiency of nuclear burning of hydrogen to helium in stars? **[5]**

**3.** (a) Explain the *Principle of General Covariance* and the importance of tensors. **[5]**

(b) Explain briefly why the Einstein equations,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

are the relativistic generalisation of Newtonian gravity allowing for spacetime curvature. **[5]**

(c) The Einstein equation for empty-space is $R_{\mu\nu} = 0$, where $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ is the Ricci tensor. Assuming a weak gravitational field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$, and the Lorentz gauge condition, $\partial_\alpha h^\alpha_\nu = \partial_\nu h/2$, show that the variations in the metric obey the wave equation

$$\Box^2 h_{\mu\nu} = 0.$$

Describe the solutions of this equation and their relation to gravitational waves. **[10]**

(d) Consider two black holes of equal mass, $M$, in circular, non-relativistic orbit of radius $a$ about their common centre of gravity with an angular speed of $\Omega = \sqrt{GM/(4a^3)}$. Assuming they orbit in the plane $(x_1, x_2, 0)$ with the origin at the centre of mass, and can be considered as point-masses, write down the coordinates of the black holes as a function of time.

Hence show that the quadrupole moment, $I^{ij}(t) = c^2 \int d^3x\, \rho(\boldsymbol{x}, t) x^i x^j$, for the black hole is given by

$$I^{ij}(t) = Mc^2a^2 \begin{pmatrix} 1 + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & 1 + \cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the density of black holes can be written using Dirac -delta functions. **[3]**

(e) Given that gravitational waves are generated by a changing quadruple in the matter distribution,

$$h^{ij}(\boldsymbol{x}, t) = -\frac{2G}{c^6 r} \left[ \frac{d^2 I^{ij}(t')}{dt'^2} \right]_{t'=t-r/c},$$

where the term in the square brackets is evaluated at the retarded time, $t'$, and $r$ is the distance to the source, derive the expression for the gravitational wave generated by the two black holes.

If the orbital radius of the black holes is slowly changing explain why the LIGO experiment could estimate the orbital radius $a$, the mass $M$, and the distance, $r$, to the black holes from gravitational waves. **[2]**