College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2015
9.30a.m. - 11.30a.m.

Chairman of Examiners
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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

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1. (a) What are the *Weak* and *Strong Equivalence Principles*? How does GR explain the observed equivalence of inertial and gravitational mass? **[4]**

(b) Use the Weak Equivalence Principle to show that the equation of motion for a freely moving particle in an arbitrary gravitational field is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,$$

explaining carefully what each terms is. **[5]**

(c) What is the *Correspondence Principle*? Given the relation

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\eta}\left(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\eta\mu} - \partial_\eta g_{\mu\nu}\right),$$

show that for a non-relativistic particle moving in a weak, time-independent gravitational field with a Newtonian potential, Φ, the acceleration of the particle reduces to $d^2\boldsymbol{x}/dt^2 = -\boldsymbol{\nabla}\Phi$, and $g_{00} = 1 + 2\Phi/c^2$. **[6]**

(d) Explain why in a weak gravitational field a distant observer will see a stationary clock which is closer to a massive body running slow. Why does this imply a photon sent between the clock and the observer will be gravitationally redshifted? **[5]**

(e) If the metric has a constant vectorial contribution, $g_{0i} = A_i$, show, to first order in $\boldsymbol{v}$, that this gives rise to a Lorentz-like acceleration

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\boldsymbol{\nabla}\Phi + c\,\boldsymbol{v}\times\boldsymbol{B},$$

where $(\boldsymbol{v}\times\boldsymbol{B})_i = (\partial_i A_j - \partial_j A_i)v^j$. Give a physical interpretation of this extra contribution to the acceleration and suggest a source for it. **[5]**

2. (a) The spacetime line element outside of a static spherically symmetric mass distribution with mass M and a non-zero cosmological constant, Λ, is

$$c^2 d\tau^2 = (1-\beta)c^2 dt^2 - \frac{dr^2}{1-\beta} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $\beta = r_S/r + \Lambda r^2/3$ and $r_S = 2GM/c^2$ is the Schwarzschild radius. By considering the Euler-Lagrange equations, or otherwise, derive the conservation equations $(1-\beta)\dot{t} = k$ and $r^2\dot{\varphi} = h$ for a particle moving in an equatorial orbit, with $\theta = \pi/2$, around this object. **[5]**

(b) Show that the radial motion of a massive particle in this spacetime is governed by the energy equation

$$\dot{r}^2 - \frac{2GM}{r} - \frac{\Lambda}{3}c^2 r^2 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3c^2} - \frac{\Lambda}{3}h^2 = c^2(k^2-1).$$

Explain what the effective potential is and describe the orbits for a massive particle and a photon when $\Lambda = 0$. **[10]**

(c) Show that the orbits for a massive particle depend on the value of Λ, but the photon orbits are independent of Λ. **[5]**

(d) The line element for a universe with linear time-independent perturbations is,

$$c^2 d\tau^2 = (1 + 2\Phi/c^2)c^2 dt^2 - (1 - 2\Phi/c^2)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$

where Φ is a spatially varying Newtonian potential. A photon travelling along a radial path will be gravitationally deflected by the potential field. Using the Euler-Lagrange equations, or otherwise, and assuming the deflection is in the equatorial plane, show to first order that the angular equation of motion of the photon is

$$\frac{d}{dt}r^2\frac{d}{dt}\varphi = 2\frac{\partial}{\partial\varphi}\Phi.$$

Explain why there is a factor of two in the deflection angle and why for a slow-moving particle this would be unity. Why is the full distortion of images on the sky caused by gravitational lensing a 2-D potential theory? **[5]**

3. (a) What is the *Principle of General Covariance*? **[5]**

(b) The stress-energy tensor for a perfect fluid with density ρ, pressure p and 4-velocity u^μ,

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu},$$

is divergence-free in Special Relativity; $\partial_\nu T^{\mu\nu} = 0$. What conservation laws does this represent? How does the stress-energy tensor and its divergence generalise to a non-inertial frame such as in a gravitational field? **[5]**

(c) Explain briefly why the Einstein equation,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} + \Lambda g^{\mu\nu},$$

is the relativistic generalisation of Newtonian gravity which allows for the curved spacetime. Explain the presence of the cosmological constant in the last term. **[5]**

(d) The line element for a traversable wormhole linking two asymptotically flat spacetimes is

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - (r^2 + r_0^2)(d\theta^2 + \sin^2\theta d\varphi^2),$$

where we extend $-\infty \le r \le \infty$. Find the equations of motion of a particle moving near the wormhole and show the non-zero affine connections are ${\Gamma^r}_{\theta\theta} = -r$, ${\Gamma^r}_{\varphi\varphi} = -r\sin^2\theta$, ${\Gamma^\theta}_{r\theta} = {\Gamma^\varphi}_{r\varphi} = r/(r^2 + r_0^2)$, ${\Gamma^\theta}_{\varphi\varphi} = -\cos\theta\sin\theta$, ${\Gamma^\varphi}_{\theta\varphi} = \cot\theta$. **[5]**

(e) Find the Ricci tensor and Ricci scalar for this spacetime, given that the Riemann tensor for spacetime curvature is

$${R^\alpha}_{\mu\beta\nu} = \partial_\beta {\Gamma^\alpha}_{\nu\mu} - \partial_\nu {\Gamma^\alpha}_{\beta\mu} + {\Gamma^\alpha}_{\beta\eta}{\Gamma^\eta}_{\mu\nu} - {\Gamma^\alpha}_{\nu\eta}{\Gamma^\eta}_{\mu\beta}.$$

Using only the $t-t$ and $r-r$ components of the Einstein equations with $\Lambda = 0$, and assuming the fluid is static ($u_0 = c$, $u_r = 0$), find the density and pressure of the fluid that would support the wormhole. What is odd about the density? **[5]**