College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2014
9.30a.m. - 11.30a.m.

Chairman of Examiners
Prof J A Peacock

External Examiner
Prof C. Tadhunter

Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

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1. (a) Define the Weak and Strong Equivalence Principles. Explain why the Equivalence Principle implies the curvature of spacetime. Why are coordinate transformations so fundamental to General Relativity? **[5]**

(b) Starting from the Strong Equivalence Principle, show that the equation of motion for a freely moving particle obeys the geodesic equation,

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

Explain each of the terms in this expression. **[4]**

(c) By considering the transformation properties of the metric, $g_{\mu\nu}$, between a general coordinate system and the local inertial frame for the relativistic line element, $c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$, show that it is a tensor. By considering the normal derivative of the metric tensor, show that

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\eta}\left(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\eta\mu} - \partial_\eta g_{\mu\nu}\right).$$

[5]

(d) Explain what is meant by the slow motion and weak field limits, and the Correspondence Principle. Assuming the gravitational field is also time-independent, show that the Newtonian equation of motion is recovered in these limits, and that

$$g_{00} = 1 + 2\frac{\Phi}{c^2},$$

where Φ is the Newtonian gravitational potential. **[6]**

(e) Global Positioning Satellites (GPS) emit a steady signal which can be used to measure the distance to a receiver. Assuming a satellite is in a geostationary orbit above the Earth, find the fractional time delay between the satellite and receiver. What would the rotation period of the Earth have to be to cancel the gravitational time delay? **[5]**

2. (a) The Reissner-Nordström line element for the spacetime around a charged, static spherically symmetric black hole of mass M and charge Q is

$$c^2 d\tau^2 = (1-\beta)c^2 dt^2 - \frac{dr^2}{1-\beta} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where $\beta = r_S/r - r_Q^2/r^2$, and we define $r_S = 2GM/c^2$ and $r_Q^2 = GQ^2/(4\pi\epsilon_0 c^4)$. Write down the Lagrangian-squared for a massive particle moving in the orbit of the black hole. **[2]**

(b) By considering the Euler-Lagrange equations, derive the conservation equations, $(1-\beta)\dot{t} = k$ and $r^2\dot{\varphi} = h$, for a particle moving in an equatorial orbit $(\theta = \pi/2)$ around this black hole. What are the conserved quantities in these relations? **[4]**

(c) Show that the radial motion of a massive particle in this spacetime is governed by the energy equation

$$\dot{r}^2 - \frac{2GM}{r} + \frac{GQ^2}{4\pi\epsilon_0 c^2 r^2} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3c^2} + \frac{GQ^2h^2}{4\pi\epsilon_0 c^4 r^4} = c^2(k^2-1).$$

By comparing with a Newtonian system, describe what each term means. For an uncharged black hole, when $Q = 0$, sketch the effective potential and possible orbits for a massive particle. **[8]**

(d) Sketch the effective potential for a massive particle on a radial trajectory $(h = 0)$ into a charged Reissner-Nordström black hole, and find the zero crossing and turning point of the potential in terms of r_S and r_Q. **[6]**

(e) By considering the time and radial components of the Reissner-Nordström line element show that there are generally two coordinate singularities, at

$$r_\pm = \frac{1}{2}\left(r_S \pm \sqrt{r_S^2 - 4r_Q^2}\right),$$

when $r_S \geq 2r_Q$. For what charge do these coordinate singularities coincide and how does the radius of the singularity relate to the turning point and zero crossing of the effective potential? Given that this coordinate singularity is an event horizon, speculate about the fate of an unbound neutral particle that crosses the event horizon along a radial path. **[5]**

3. (a) Explain what the principle of *General Covariance* is. Define tensors and explain, without mathematical detail, what the covariant derivative, ∇_μ, is. Explain the three rules which allow us to write down the equations of physics in a gravitational field. **[5]**

(b) In a local inertial frame the electromagnetic field tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu = (\phi, \boldsymbol{A})$ is the electromagnetic 4-vector potential. Show that this form for $F_{\mu\nu}$ remains valid in a gravitational field. Show that the field tensor obeys the cyclic relation $\partial_\eta F_{\mu\nu} + \partial_\mu F_{\nu\eta} + \partial_\nu F_{\eta\mu} = 0$, and write down this relation in a gravitational field. Write down the electromagnetic field equation $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, where J^ν is the 4-vector current, in a gravitational field. **[7]**

(c) Explain briefly why the Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

is the relativistic generalisation of Newtonian gravity which allows for the curved spacetime. The Riemann tensor in a local inertial frame is $R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu}$. Show that the derivate of this, $\partial_\eta R^\alpha{}_{\mu\beta\nu}$, obeys a cyclic relation in the indices η, β and ν. Write down this relation in a gravitational field and, given $R_{\alpha\mu\beta\nu} = -R_{\alpha\mu\nu\beta}$ and $R_{\alpha\mu\beta\nu} = R_{\mu\alpha\beta\nu}$, show that $\nabla_\alpha R^\alpha_\mu = \nabla_\mu R$. Hence explain why

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

is the correct form for the Einstein tensor. **[8]**

(d) A consistent, special relativistic theory of gravity based only on a scalar field can be written down as,

$$\Box^2 \Phi = -\frac{4\pi G}{c^2} T,$$

where $\Box^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$, the stress-energy scalar is $T = T^\mu{}_\mu$, and Φ is the gravitational potential. Show that this recovers Newtonian gravity in the non-relativistic limit. Discuss how such a theory be ruled out on fundamental theoretical and experimental grounds. **[5]**