THE UNIVERSITY OF EDINBURGH

College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2013
9.30a.m. - 11.30a.m.

Chairman of Examiners
Prof J A Peacock

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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

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1. (a) Explain the terms *weak equivalence principle* and *strong equivalence principle*. Explain the relation of these principles to the equality of gravitational and inertial mass. **[5]**

(b) By transforming from a local inertial frame, ξ^α, to an arbitrary coordinate system, x^μ, show that the strong equivalence principle implies a free-moving particle obeys the geodesic equation:

$$\dot{u}^\lambda + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0.$$

Explain each of the terms in this expression. **[4]**

(c) The affine connection is not a tensor. Show that under a general change of coordinates it transforms as

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \Gamma^\rho_{\alpha\beta} + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}.$$

[5]

(d) By considering the geodesic equation in the stationary, weak-field, slow-motion limit show that it reproduces the Newtonian equation of motion,

$$\ddot{x}_i = -\nabla_i \Phi.$$

Explain how the *correspondence principle* is used to relate the 00-component of the metric to the Newtonian potential, Φ. **[6]**

(e) Consider a satellite in circular orbit around the Earth, at a radius of r from the centre of the Earth, sending signals to a receiver based at the South Pole. Show that the ratio of the time measured at the receiver to that recorded on the satellite is

$$\frac{\Delta\tau_{\text{receiver}}}{\Delta\tau_{\text{satellite}}} = 1 - \frac{GM_E}{R_E c^2} + \frac{3GM_E}{2rc^2},$$

where R_E and M_E are the radius and mass of the Earth. How far away does the satellite have to be before the time on the satellite runs faster than the receiver? **[5]**

2. (a) A black hole of mass M has the line element

$$c^2 d\tau^2 = (1-\beta)c^2 dt^2 - \frac{dr^2}{1-\beta} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $\beta = r_S/r$ and $r_S = 2GM/c^2$ is the Schwarszchild radius. Write down the Lagrangian-squared for a particle moving in the orbit of the black hole. **[2]**

(b) Derive the conservation equations, $(1-\beta)\dot{t} = k$ and $r^2\dot{\phi} = h$, for a particle moving in an equatorial orbit around a black hole. What is the angular velocity of the particle, $\Omega = d\phi/dt$, seen by an observer at infinity? What is the apparent angular velocity of the particle as it crosses the Schwarszchild radius? **[8]**

(c) Show that the radial motion of a massive particle is governed by

$$\dot{r}^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3c^3} = c^2(k^2-1).$$

Sketch the effective potential for this system and describe the motion of the particle when $h = \sqrt{3}r_S c$. **[8]**

(d) The Kerr metric for a rotating black hole was discovered 50 years ago, in 1963. A particle moving in the equatorial plane ($\theta = \pi/2$) of a black hole rotating with angular moment, J, can be written as

$$c^2 d\tau^2 = c^2 dt^2 - \beta(cdt - ad\phi)^2 - \frac{dr^2}{(1-\beta+a^2/r^2)} - (r^2+a^2)d\phi^2,$$

where $a = J/Mc$. Write down the Lagrangian-squared and find the conservation equation for ϕ. Show that the apparent angular motion of an in-falling particle with no initial angular momentum, seen by an observer at infinity, is

$$\Omega = \frac{ac\, r_S}{r^3 + a^2(r + r_S)}.$$

What is the apparent angular velocity as it passes $r = r_S$? **[7]**

3. (a) What is the *principle of general covariance* and how it is used to find the equations of physics in a gravitational field? Define tensors and covariant derivatives and explain how they are used to satisfy this principle. **[4]**

(b) What is *parallel transport*? By considering the parallel transport of a vector, v^α, around a small rectangle of sides δa^μ and δb^β, show that the change in the vector is

$$\delta v^\alpha = R^\alpha{}_{\nu\beta\mu} v^\nu \delta a^\mu \delta b^\beta,$$

where $R^\alpha{}_{\nu\beta\mu}$ is the Riemann tensor. **[8]**

(c) The Einstein field equation are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $T^{\mu\nu}$ is the stress-energy tensor for a perfect fluid and

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\alpha_{\nu\eta}\Gamma^\eta_{\mu\beta},$$

is the Riemann tensor. Show that in the Newtonian limit of a stationary, weak field and slow-motion, the 00-component of the Einstein field equations reduces to the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho.$$

[8]

(d) Consider the off-diagonal term G_{0i} of the Einstein field equations. Allowing for an off-diagonal term in the metric, $g_{0i} = h_{0i} = A_i/c$, where A_i is a vector-potential, and the stress-energy tensor is $T_{0i} = \rho u_i u_0$, show that gravity can be re-cast in the form of Maxwell's equations for gravity,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 4\pi G \rho, & \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= 0, & \boldsymbol{\nabla} \times \boldsymbol{B} &= -\frac{16\pi G}{c^2} \boldsymbol{j} \end{aligned}$$

where $\boldsymbol{E} = \boldsymbol{\nabla}\Phi$, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{j} = \rho \boldsymbol{u}$. You may wish to use the vector identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$. **[5]**