College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2012
9.30a.m. - 11.30a.m.

Chairman of Examiners
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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

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1. (a) Describe how the equivalence of inertial and gravitational mass leads to the *Weak Equivalence Principle*. What is the *Strong Equivalence Principle*? **[3]**

(b) By transforming from a local inertial coordinate system, ξ^α, in which

$$c^2 d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

to a general coordinate system, x^μ, show that the Strong Equivalence Principle implies a particle in free-fall obeys the geodesic equation:

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0,$$

where $\dot{x}^\mu \equiv dx^\mu/d\tau$ and

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}.$$

is the affine connection. **[5]**

(c) Show how the metric transforms between a local inertial frame and the general coordinate system. Given the Lagrangian-squared is

$$L^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu,$$

use the Euler-Lagrange equation, or otherwise, to show that the affine connection is

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\eta}(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\eta\mu} - \partial_\eta g_{\mu\nu}).$$

Explain the role of the affine connection in the geodesic equation. **[7]**

(d) Defining the 4-velocity $u^\lambda \equiv \dot{x}^\lambda$, re-write the geodesic equation as a force equation in terms of u^λ. By transforming to the co-vector, $u_\mu = g_{\mu\lambda}u^\lambda$, derive the equation of motion

$$\dot{u}_\mu = \frac{1}{2}(\partial_\mu g_{\lambda\nu})u^\lambda u^\nu.$$

Hence, argue that if the metric is stationary with respect to the particles proper time, $\dot{g}_{\mu\nu} = 0$, the equation of motion in terms of the vector u^λ has a similar form. **[5]**

(e) By using the results of part (d), or otherwise, show that in the slow-motion, $u^i \ll u^0 \approx c$, weak-field limit, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, and the metric is stationary, $\dot{g}_{\mu\nu} = 0$, the Newtonian equation of motion for a particle is recovered with

$$h_{00} = \frac{2\Phi}{c^2},$$

where Φ is the Newtonian gravitational potential. **[5]**

2. (a) The line-element around a black hole, of mass M, is

$$c^2 d\tau^2 = \alpha c^2 dt^2 - \frac{dr^2}{\alpha} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right),$$

where $\alpha \equiv 1 - 2GM/(rc^2)$. Write down the Lagrangian-squared for a particle moving in the orbit of the black hole. **[2]**

(b) Using the Euler-Lagrange equations, or otherwise, show that $\alpha\dot{t} = k$ and $r^2\dot{\phi} = h$, for a particle moving in an equatorial orbit, where k and h are constants and a dot indicates derivative with respect to an affine parameter. What do these equations express? **[5]**

(c) Show that the radial motion of a massive particle is governed by

$$\dot{r}^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3c^3} = c^2(k^2 - 1).$$

Explain what each of these terms refers to. **[6]**

(d) Define an effective potential for particle moving in the gravitational field of a black hole. By sketching the effective potential for different angular motions of a massive particle, sketch and describe the possible orbits of a massive particle around the black hole in the $r - \phi$ plane, and under what conditions it will fall into the black hole. **[4]**

(e) The line element around static, infinitely long, thin, cylindrically symmetric cosmic string in cylindrical coordinates, (t, r, θ, z), is

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2(1 - 4G\mu/c^2)^2 d\theta^2 - dz^2,$$

where G is Newtons constant and μ is the string tension. Assuming a particle moves in the plane $z = 0$, use the Euler-Lagrange equation to show that $\dot{t} = k$ and $r^2\dot{\theta} = h$ for constants k and h. Show for a massive particle moving past the cosmic string that the radial energy equation is the same as for flat-space, with a modified angular term. **[4]**

(f) Solve the radial energy equation to show that a passing massive particle will be deflected by an angle $\Delta\theta = 4\pi G\mu/c^2$. Argue that this will be the same for a massless particle and so the cosmic string will act like a lens. Finally, argue that the space around the cosmic string is flat but with an angle of $8\pi G\mu/c^2$ carved out of it. **[4]**

3. (a) Explain what the *principle of general covariance* is, and how it is used to find the equations of physics in a gravitational field. What is the *correspondence principle* and why do we need it? **[4]**

(b) The Riemann tensor is

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\beta\eta}\Gamma^{\eta}_{\mu\nu} - \Gamma^{\alpha}_{\nu\eta}\Gamma^{\eta}_{\mu\beta}$$

Show that in a local inertial frame this tensor obeys the relation

$$\partial_\eta R^{\alpha}{}_{\mu\beta\nu} + \partial_\beta R^{\alpha}{}_{\mu\nu\eta} + \partial_\nu R^{\alpha}{}_{\mu\eta\beta} = 0.$$

Hence argue that in a general coordinate frame this leads to the *Bianchi Identity*

$$\nabla_\eta R^{\alpha}{}_{\mu\beta\nu} + \nabla_\beta R^{\alpha}{}_{\mu\nu\eta} + \nabla_\nu R^{\alpha}{}_{\mu\eta\beta} = 0,$$

where ∇_μ is the covariant derivative. **[6]**

(c) Given that the Riemann tensor is anti-symmetric in its last two indices, and antisymmetric in its first two indices, show that

$$\nabla_\mu R = 2\nabla_\alpha R^{\alpha}{}_{\mu}.$$

Hence, show that the covariant divergence of the Einstein tensor

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$$

is zero. **[6]**

(d) Explain what the stress-energy tensor, $T^{\mu\nu}$, is and argue without detailed mathematics why the Einstein equation,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

is the relativistic generalisation of the Poisson equation. **[4]**

(e) A toy model for exploring quantum gravity is a 2-D spacetime with one time direction and one space direction. Assuming the metric for a uniform, expanding 2-D spacetime containing a fluid with constant mass-density and zero pressure is

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t) dr^2,$$

find the affine connections for this model from the Euler-Lagrange equations (or otherwise). Hence calculate the Ricci tensor, $R_{\mu\nu}$ and the Einstein tensor, $G_{\mu\nu}$,. Explain why this 2-D toy model must be empty. **[5]**