College of Science and Engineering
School of Physics

General Relativity

SCQF Level 11, U01429, PHY-5-GenRel

???, 2011
9.30a.m. - 11.30a.m.

Chairman of Examiners
Prof J A Peacock

External Examiner
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Answer **TWO** questions

The bracketed numbers give an indication of the value assigned to each portion of a question.

Only the supplied Electronic Calculators may be used during this examination.

Anonymity of the candidate will be maintained during the marking of this examination.

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1. (a) What is the *Strong (Einstein) Equivalence Principle*? **[2]**

(b) By considering a freely-moving particle in a gravitational field, show that the Equivalence Principle leads to the geodesic equation:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,$$

where

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}.$$

[5]

(c) By considering the change in frames implied by the Equivalence Principle, or otherwise, show that the affine connection is related to the metric tensor, $g_{\mu\nu}$, by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\eta}\left(\partial_\mu g_{\eta\nu} + \partial_\nu g_{\eta\mu} - \partial_\eta g_{\mu\nu}\right),$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. **[5]**

(d) Show that in the stationary, weak-field limit, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, the Newtonian equation of motion for a particle is recovered and that

$$h_{00} = 2\Phi/c^2,$$

where Φ is the Newtonian gravitational potential. **[5]**

(e) Explain in this weak-field limit why an observer at large distances from a massive body will see a stationary clock near to the body running slow. Why does this imply there will be a gravitational redshifting of light sent between two stationary observers in the gravitational field? Illustrate the effect of time dilation seen in the stationary frame. **[5]**

(f) Keeping terms which are first-order in v/c, assuming the metric is stationary and $h_{0i} = B_i$ is non-zero, show the equation of motion for a test particle is

$$\ddot{x}_i \approx -\partial_i\Phi + c(\partial_j B_i - \partial_i B_j)v^j.$$

[3]

2. (a) The Schwarzschild line element for a compact, spherically symmetric mass is

$$c^2 d\tau^2 = \alpha c^2 dt^2 - \frac{dr^2}{\alpha} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $\alpha \equiv 1 - 2GM/(rc^2)$. Write down the Lagrangian-squared, L^2, for a particle moving in this spacetime. **[2]**

(b) Using the Euler-Lagrange equations, or otherwise, show that $\alpha\dot{t} = k$ and $r^2\dot{\phi} = h$, for a particle moving in an equatorial orbit, where k and h are constants and a dot indicates derivative with respect to an affine parameter. What do these equations express? **[5]**

(c) Hence show that the radial motion of a massive particle is governed by

$$\dot{r}^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3c^3} = c^2(k^2 - 1).$$

Explain what each of these terms refers to. **[6]**

(d) For a photon travelling on a radial trajectory towards a black hole show that

$$\dot{r}^2 = c^2k^2.$$

Hence, using $\dot{t} = k/\alpha$, show that the apparent velocity of the photon seen by a distant observer is $dr/dt = \pm\alpha c$. **[5]**

(e) Integrate the photon velocity to find t as a function of coordinate radius in the regime $r \gg r_s$ and $r \ll r_s$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. Sketch these trajectories in the $t - r$ coordinate plane of a distant observer and indicate what happens to light-cones in each regime. **[5]**

(f) Describe what a distant observer would see as objects fall into black hole. **[2]**

3. (a) What are the *correspondence principle* and the *principle of general covariance*? **[4]**

(b) In Special Relativity the tensor describing a perfect fluid of density ρ, pressure p and 4-velocity $U^\mu = dx^\mu/d\tau$, is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - p\eta^{\mu\nu}$$

and has zero 4-divergence, $\partial_\nu T^{\mu\nu} = 0$.

The 4-velocity satisfies the normalisation condition $U^\nu U_\nu = c^2$. Show the differential of this condition, with respect to $\partial/\partial x^\mu$, can be used to find the conservation of matter current, $j^\mu = \rho U^\mu$, for a pressureless fluid, starting from the expression $U^\nu \partial_\mu T^{\mu\nu} = 0$. **[5]**

(d) Using the principle of general covariance explain how the SR conservation of energy and momentum can be generalised to $\nabla_\nu T^{\nu\mu}$ to be valid in the non-inertial frames considered in General Relativity. Explain what the operator ∇_ν means here. **[3]**

(e) Argue, without detailed mathematics, why the Einstein equations may be written

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

[3]

(f) Given the Riemann tensor is

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\alpha_{\nu\eta}\Gamma^\eta_{\mu\beta}$$

and its contraction is $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, show the Einstein equations can be reduced to the Poisson equation of Newtoniran gravity in the weak-field limit, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, and in the slow-motion regime where $v \ll c$. Note that $\Gamma^i_{00} = \delta^{ij}\partial_j h_{00}/2$ and $U_0 = c$ in this limit. **[6]**

Show in the weak-field limit that the Riemann tensor can be written in terms of a small perturbation in the metric tensor, $h_{\mu\nu}$, as

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(\partial_\beta\partial_\mu h_{\nu\alpha} - \partial_\nu\partial_\mu h_{\beta\alpha} + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\beta\partial_\alpha h_{\mu\nu}).$$

Hence contract this to find $R_{\mu\nu}$ and R. Using the Lorentz Gauge where $\partial_\mu h^{\mu\nu} = \eta^{\nu\mu}\partial_\mu h$, show the linearized Einstein equation is

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\left(h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\right) = -\frac{16\pi G}{c^4}T_{\mu\nu}.$$

[4]