PHYS11010: General Relativity 2024–2025

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http://www.roe.ac.uk/japwww/teaching/gr.html

Synopsis

This is a course on General Relativity for physicists, designed to develop and apply the mathematical tools involved in curved spacetime, starting from a foundation of physical principles. The aim is to emphasise that General Relativity is at heart a simple and intuitive theory, which generalises familiar elements such as the constant speed of light in special relativity, and the universality of free fall. The language of tensor analysis is developed in order to write covariant equations of physics, which apply in a coordinate-free manner. The key equations of this sort are the geodesic equation, which generalises Newton's 2nd law, and Einstein's field equations, which generalise Poisson's equation. These tools are applied to weak gravitational fields: planetary orbits and their post-Newtonian precession; gravitational deflection of light; gravitational waves as time-dependent distortions of spacetime. Strong gravitational fields are considered in the context of black holes and isotropic cosmological models.

Textbooks

These notes are intended to be self-contained, but there are many excellent textbooks on the subject. The following are especially recommended for background reading:

- Hobson, Efstathiou & Lasenby (Cambridge): *General Relativity: An introduction for Physicists*. This is fairly close in level and approach to this course.
- Ohanian & Ruffini (Cambridge): *Gravitation and Spacetime (3rd edition)*. A similar level to Hobson et al. with some interesting insights on the electromagnetic analogy.
- Cheng (Oxford): *Relativity, Gravitation and Cosmology: A Basic Introduction*. Not that 'basic', but another good match to this course.
- D'Inverno (Oxford): *Introducing Einstein's Relativity*. A more mathematical approach, without being intimidating.
- Weinberg (Wiley): *Gravitation and Cosmology*. A classic advanced textbook with some unique insights. Downplays the geometrical aspect of GR.
- Misner, Thorne & Wheeler (Princeton): *Gravitation*. The classic antiparticle to Weinberg: heavily geometrical and full of deep insights. Rather overwhelming until you have a reasonable grasp of the material.

It may also be useful to consult background reading on some mathematical aspects, especially tensors and the variational principle. Two good references for mathematical methods are:

- Riley, Hobson and Bence (Cambridge; RHB): *Mathematical Methods for Physics and Engineering*
- Arfken (Academic Press): *Mathematical Methods for Physicists*

Contents

1 Overview

General Relativity (GR) has an unfortunate reputation as a difficult subject, going back to the early days when the media liked to claim that only three people in the world understood Einstein's theory. But while there are occasional mathematical challenges to negotiate, GR is in many ways one of simplest and most natural parts of physics, where the mathematical aspects emerge from a foundation built on simple but powerful intuitive insights.

GR is a completion of the logic of Special Relativity (SR), which states that there is no preferred standard of rest, so that space and time as experienced locally by all different observers must provide equally valid descriptions of the universe. As a consequence, Newton's absolute space and time must be abandoned. SR considers only observers moving at constant velocity; but if there is no absolute standard of motion, then surely the viewpoint of *all* observers should be equally valid? The core aim of GR is therefore to show how the laws of physics can be set up in a way that is completely independent of the state of motion of the observer. We will see that this leads to the three main elements that characterise GR:

(1) GR is a theory of spacetime as experienced by (in general) accelerated observers;

(2) As a consequence, spacetime must be curved, and the curvature affects particle trajectories;

(3) GR is also a relativistic theory of gravity, but where matter influences curvature rather than just determining a Newtonian gravitational force.

All this was memorably captured by John Archibald Wheeler: *"matter tells spacetime how to curve and curved spacetime tells matter how to move"*. From these fundamental insights, we obtain an impressive list of applications: gravitational time dilation; velocity-dependent gravitational forces; gravitational deflection of light; gravitational waves; black holes, where spacetime becomes singular; and the spacetime of the expanding universe. This course will touch on all of these topics.

2 Elements of Special Relativity

2.1 4-vectors and the Lorentz transformation

We start with a brief review of Special Relativity, aiming to set the scene for what follows. The key concept is that we are concerned with **events in spacetime**, and in particular the **spacetime interval** between them. This can be written as a **4-vector**:

$$dx^\mu = (c\,dt, dx, dy, dz) \quad \mu = 0, 1, 2, 3. \tag{1}$$

The interval does not have to be infinitesimal, but it will often be convenient to focus on this case. This vector has a **norm**, which is a quantity that is independent of reference frame, i.e. is the same for all observers – known as an **invariant**. This is obtained by defining another vector with the index 'downstairs':

$$dx_\mu = (c\,dt, -dx, -dy, -dz), \tag{2}$$

so that the squared norm is

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dx^\mu dx_\mu = \eta_{\mu\nu}\, dx^\mu dx^\nu. \tag{3}$$

The matrix $\eta_{\mu\nu}$ is $\mathrm{diag}(1, -1, -1, -1)$; note the use of the summation convention on repeated indices, as usual. We have written the invariant in terms of $d\tau$, the **proper time interval**, which clearly means the time interval between two events at the same spatial location – it is just the ticking of the clock that the observer carries along with them.

The interval is zero or **null** for events that are connected by light signals, and one of the key steps to SR was requiring that this should hold for all observers, so that the speed of light is just a property of empty space, which all observers experience in equivalent ways (as was proved empirically by the Michaelson–Morley experiment, but Einstein considered the result inevitable). This looks implausible at first: if we make a **Galilean transformation** for an observer that moves at v in the x direction and write $t' = t$, $x' = x - vt$, then $dx'/dt' = dx/dt - v$ and so it seems that the speed of light alters. The solution is that this Newtonian approach must be wrong, and we assume instead that the spacetime intervals measured by different observers have a more general linear relation:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, dx^{\nu} \equiv \Lambda^{\mu}_{\nu}\, dx^{\nu}. \tag{4}$$

It is then not too hard to show that the requirement of constant c leads to the transformation matrix of the **Lorentz transformation**:

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{5}$$

where $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$ and the boosted observer is assumed to move at velocity v along the x direction. Given this matrix, it is easy to verify that $dx^{\mu}dx_{\mu}$ is unchanged by a Lorentz transformation, so that the proper time interval is indeed a relativistic invariant, as asserted above.

Why are we going to the trouble of having two different kinds of vectors? This does not appear in the familiar treatment of vectors in Euclidean space, but this is only because we are usually able to define the components of vectors using a set of basis vectors that are orthonormal. But this doesn't have to be the case, and we can use a **skew basis**. We can express any vector as a linear superposition:

$$\mathbf{V} = \sum_i V_i^{(1)}\, \mathbf{e}_i, \tag{6}$$

where $\mathbf{e}_i$ are the basis vectors. But we are used to extracting components by taking the dot product, so we might equally well want to define a second kind of component by

$$V_i^{(2)} = \mathbf{V} \cdot \mathbf{e}_i. \tag{7}$$

These numbers are not the same, as can be seen by inserting the first definition into the second: if the basis is not orthogonal, then a given type-2 component is a mixture of all the different type-1 components. The two types of component are named respectively **contravariant** and **covariant** components, and both are required in order to get the modulus-squared of a vector:

$$V^2 = \mathbf{V} \cdot \mathbf{V} = \mathbf{V} \cdot \left(\sum_i V_i^{(1)}\, \mathbf{e}_i \right) = \sum_i V_i^{(1)}\, V_i^{(2)}. \tag{8}$$

Similarly, $\mathbf{A} \cdot \mathbf{B} = \sum_i A_i^{(1)}\, B_i^{(2)} = \sum_i A_i^{(2)}\, B_i^{(1)}$. This is exactly the manipulation we needed in order to obtain invariants in SR.

A common example of a covariant 4-vector is the 4-derivative: $\partial_\mu \equiv \partial/\partial x^{\mu}$: having the upstairs index downstairs amounts to having a downstairs index. We can see that this is sensible: the change in a scalar field is $d\phi = (\partial\phi/dx^{\mu})\, dx^{\mu} = \partial_\mu \phi\, dx^{\mu}$, which will be invariant as desired (a scalar quantity is unchanged under Lorentz transformation). Explicitly, $\partial_\mu = (\partial/\partial ct, \boldsymbol{\nabla})$ and $\partial^\mu = (\partial/\partial ct, -\boldsymbol{\nabla})$, so that $\partial_\mu \partial^\mu = (1/c^2)\partial^2/\partial t^2 - \nabla^2 \equiv \Box$. The RHS here is the d'Alembertian wave operator, and its relativistic form means that a scalar function of space and time that solves the wave equation in one frame solves it in any frame. A further nice example of the 4-derivative at work is in the description of conserved quantities, such as charge. Define the 4-current in terms of the charge density and current density, $J^\mu = (c\rho, \mathbf{j})$; this allows us to write the invariant equation $\partial_\mu J^\mu = 0$, which is a compact form of the **continuity equation**, $\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$.

To obtain laws of physics that are valid in SR, we are thus naturally led to write equations in terms of 4-vectors. This reflects the **principle of general covariance**, which says that valid laws of physics should be independent of coordinates – i.e. should hold for all observers. A 4-vector law $A^\mu = B^\mu$ is naturally covariant as both sides of the equation change in the same way under Lorentz transformations: if the law holds in one frame of reference, it holds in general. Note the unfortunate historical baggage here: 'covariance' of physical laws has no direct relation to 'covariant components'.

2.2 Relativistic dynamics

A good example of this reasoning is supplied by the 4-momentum. Suppose we have a collision between a set of free particles, for which we would say there is no change in total momentum: $\Delta \sum_i \mathbf{p}_i = 0$. How can we write this in an observer-independent fashion? A natural 4-vector to construct is rest mass times 4-velocity:

$$P^\mu = mU^\mu = m\, dx^\mu/d\tau = m\,\gamma\, dx^\mu/dt = m(\gamma c, \gamma \mathbf{v}) \tag{9}$$

(note the replacement of $d/d\tau$ by $\gamma\, d/dt$; this comes from the Lorentz transformation of $dx^{\mu\prime} = (c\, d\tau, 0)$ in the rest frame, and is a common manipulation in SR). If we write $\Delta \sum_i P_i^\mu = 0$, this is true in all frames (a zero 4-vector remains zero under Lorentz transformation). For non-relativistic particles, the spatial part of this equation gives the desired $\Delta \sum_i \mathbf{p}_i = 0$, suggesting that $\gamma m \mathbf{v}$ should be identified as the momentum in general. For free, we also get a further conservation law: $\Delta \sum_i P_i^0 = 0$ – what does this correspond to physically? Since $P^0 = \gamma mc$, this looks like conservation of mass in the nonrelativistic limit. But $P^0 c \simeq mc^2 + mv^2/2$, where the 2nd term is the kinetic energy. This suggests that $P^0 c$ is a total energy, including the radical idea of an energy equivalent of rest mass.

To show this more convincingly, consider the proper time derivative of the 4-momentum, which is related to the 4-acceleration:

$$\frac{d}{d\tau}\, P^\mu = m\, A^\mu = (\gamma\, d(\gamma mc)\,/dt, \gamma d\mathbf{p}/dt). \tag{10}$$

We want to keep the usual definition of force as rate of change of momentum, so there is a 4-force, $F^\mu = (F^0, \gamma \mathbf{f})$, which can be used to write the relativistic generalization of $f = m\mathbf{a}$:

$$F^\mu = m\, A^\mu. \tag{11}$$

What is the time component of F^μ? We know it has to satisfy $F^0 = mA^0 = \gamma\, d(\gamma mc)/dt$, but this can also be written in terms of the force $\mathbf{f}$ by using the invariant $A^\mu U_\mu = 0$. This is proved in the rest frame of the particle, where $U^\mu = (U^0, 0, 0, 0)$ and noting that $d\gamma/dt = 0$ when the velocity is zero, so that $A^0 = 0$. Hence we have in general

$$\gamma c A^0 - \gamma \mathbf{v} \cdot \gamma d(\gamma \mathbf{v})/dt = 0, \tag{12}$$

implying

$$F^0 = mA^0 = (\gamma/c)\, \mathbf{v} \cdot d(\gamma m \mathbf{v})/dt = (\gamma/c)\, \mathbf{v} \cdot \mathbf{f}. \tag{13}$$

Thus the time component of $F^\mu = m\,A^\mu$ says

$$(\gamma/c)\,\mathbf{v}\cdot\mathbf{f} = \gamma\,d(\gamma mc)/dt \Rightarrow \mathbf{v}\cdot\mathbf{f} = d(\gamma mc^2)/dt, \tag{14}$$

leading us to identify γmc^2 as the total energy (because $\mathbf{v}\cdot\mathbf{f}$ is the rate at which the force does work). This is the famous $E = Mc^2$, but note that we prefer not to introduce the 'relativistic mass': m always means the rest mass. So our relation $\Delta\sum_i P_i^0 = 0$ amounts to conservation of energy: this had to arise if we try to express conservation of momentum in terms of the 4-momentum $P^\mu = (E/c, \mathbf{p})$.

2.3 Distinguishing Special and General Relativity

Reviewing the above logic, it should be apparent that most of SR does not require a restriction to observers moving at constant velocity. We have in fact already stated the basic premise of General Relativity, which is general covariance: valid laws of physics should apply for all observers, whatever their state of motion, and the way to ensure this is to write laws using 4-vectors and invariants so that both sides of any equation transform in the same way.

The only problem, then, is to figure out how to construct the desired 4-vectors. Some of what we have done goes through immediately: dx^μ is a 4-vector by definition, and proper time $d\tau$ can be seen to be an invariant on physical grounds: it is defined by the ticking of a clock in the rest frame of a particle. Thus the 4-velocity $U^\mu = dx^\mu/d\tau$ is a general 4-vector and hence so is the 4-momentum $P^\mu = mU^\mu$ (like proper time, rest mass is also an invariant on physical grounds). So our law for conservation of momentum and energy in collisions applies in general. But things go wrong at the next level, when we try to construct the 4-acceleration. Consider the transformation law for spacetime coordinates: $dx'^\mu = \Lambda^\mu_\nu dx^\nu$. Dividing by $d\tau$ shows that 4-velocity obeys the same transformation:

$$U'^\mu = \Lambda^\mu_\nu U^\nu. \tag{15}$$

But to perform dynamics we need the 4-acceleration, and this requires us to differentiate with respect to τ. If the transformation coefficients Λ^μ_ν are constants, then the differentiation goes through them and $dU^\mu/d\tau$ obeys the standard 4-vector transformation law. We can then write e.g. $A^\mu = 0$ as the relativistic generalization of Newton's first law. But for general coordinate transformations, such as boosts to the frame of an accelerating observer, there is no guarantee that Λ^μ_ν will be constant as in the Lorentz transformation. In that case, we risk the appearance of terms involving the derivatives of Λ^μ_ν, which spoils the transformation law.

Thus we see that $F^\mu = mA^\mu$ cannot be considered a generally valid relativistic law of physics. Einstein saw how to solve this problem in a tremendous leap of intuition, by thinking about the case of gravitational forces.

3 The Equivalence Principle

At the core of GR is the **Equivalence Principle**, which is an elaboration of the simple observation that objects in a gravitational field fall equally fast, independent of their mass. Although familiar since Galileo, in Einstein's hands this fact becomes the bridge to the relativistic generalisation of acceleration, and directly leads to curved spacetime.

3.1 Mass in Newtonian physics

In 1590, Galileo argued that objects made from different materials will all fall at the same rate. Around the 1670's, Newton defined force in terms of rate of change of momentum:

$$\mathbf{F} = \frac{d}{dt} m_{\rm i}\mathbf{v} = m_{\rm i}\,\ddot{\mathbf{x}}, \tag{16}$$

where $m_{\rm I}$ is the **inertial mass**. For gravity, the force on a particle is proportional to a gravitational acceleration field, $\mathbf{g}$:

$$\mathbf{F} = m_{\rm gp}\,\mathbf{g}, \tag{17}$$

where $m_{\rm gp}$ is the **passive gravitational mass**; thus

$$\ddot{\mathbf{x}} = \left(\frac{m_{\rm gp}}{m_{\rm i}}\right)\mathbf{g}. \tag{18}$$

For all objects to fall at the same rate requires that $m_{\rm gp} = m_{\rm i}$, but there is no reason to expect this in Newtonian physics.

There is another types of mass in Newtonian physics associated with gravity, the **active gravitational** mass:

$$\mathbf{g} = -\frac{G m_{\rm ga}}{r^2}\,\hat{\mathbf{r}}. \tag{19}$$

Newton's 3rd law ensures that $m_{\rm ga} = m_{\rm gp}$, since $\mathbf{F}_{12} = -\mathbf{F}_{21} \Rightarrow G m_{\rm ga1} m_{\rm gp2}/r^2 = G m_{\rm ga2} m_{\rm gp1}/r^2$, and hence

$$\frac{m_{\rm ga1}}{m_{\rm gp1}} = \frac{m_{\rm ga2}}{m_{\rm gp2}} \tag{20}$$

and the value of G can be adjusted so that the active and gravitational masses are not just proportional, but equal. Henceforth we shall simply write $m_{\rm gp} = m_{\rm ga} = m_{\rm g}$.

3.1.1 Eötvös experiments ($\sim$ 1890)

The difference between inertial and gravitational mass can be measured using **Torsion Balance** experiments. These torsion balance experiments use masses of different materials, but the same gravitational mass $m_{\rm g}$ (which can be checked with a spring balance). The force on each is the sum of the gravitational force and the force from the fibres making up the balance. But these forces do not match exactly, because the balance rotates around the Earth, with a weak acceleration $\mathbf{a}$, so the difference in force must supply $m_{\rm i}\mathbf{a}$ for each mass. But if the inertial masses are different, then these differences are not equal and the torsion balance must rotate a little in compensation. Because $\mathbf{a}$ changes with period 24h, the balance would then have to oscillate with the same period. From the lack of such oscillations, Eötvös and his team showed that gravitational and inertial masses were equal to 1 part in 20 million. Present-day experiments of this type allow variations in the ratio of no larger than about 10^{-13}.

3.2 Inertial frames and inertial forces

A further puzzle regarding Newtonian mechanics is that $\mathbf{F} = m\mathbf{a}$ only applies in inertial frames of reference. What exactly are these? The definition is circular, with inertial frames being defined as involving those sets of observers for whom $\mathbf{F} = m\mathbf{a}$ applies. But having somehow found an inertial frame, it is easy to generate an non-inertial one by considering the point of view of an observer who accelerates relative to that frame. As is well known, additional so-called **fictitious forces** or **inertial forces** then appear in the equation of motion. In respectively linearly accelerating and rotating frames, we would write

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} + m\mathbf{g} \\ \mathbf{F} &= m\mathbf{a} + m\boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\mathbf{r}) - 2m(\mathbf{v}\wedge\boldsymbol{\Omega}) + m\dot{\boldsymbol{\Omega}}\wedge\mathbf{r}. \end{aligned} \tag{21}$$

The latter expression adds the **Euler force** to the familiar centrifugal and Coriolis forces.

All physicists are taught at school that these extra forces are not real, but this should make us deeply unhappy from a relativistic point of view. If there is no absolute rest, why shouldn't the viewpoint of an accelerating observer be valid? No physicist should be happy to say that although such observers see forces, these have no known cause. The relativist's attitude will be that if our physical laws are correct, they should account for what observers see from any arbitrary point of view. The 'fictitious' forces must be real – as is well-known to anyone who has ever experienced them.

The mystery of inertial frames is deepened when we note that an inertial frame is one in which the bulk of matter in the universe is at rest. This observation was taken up in 1872 by Ernst Mach. He argued that since the acceleration of particles can only be measured relative to other matter in the universe, the existence of inertia for a particle must depend on the existence of other matter. This idea has become known as **Mach's Principle**, and was a strong influence on Einstein in formulating general relativity. In fact, Mach's ideas ended up very much in conflict with Einstein's eventual theory – most crucially, the rest mass of a particle is a relativistic invariant, independent of the gravitational environment in which a particle finds itself.

But for the present purpose, the key observation is that the inertial forces are proportional to mass, just as gravitational force is. This suggests the following powerful insight:

Perhaps fictitious forces can be understood as gravitational effects – or, equivalently, gravity is a fictitious force too.

3.3 Inertial forces can be transformed away

Following this insight, and noting that inertial forces appear via the transformation to an accelerating frame, we can see that a suitable transformation can also remove such forces – including gravity.

Consider a particle inside a freely-falling box in a gravitational field $\mathbf{g}$: its equation of motion is

$$m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{g} + \mathbf{F}, \tag{22}$$

where $\mathbf{F}$ represents non-gravitational forces. Now move to the rest frame of the box, i.e. make the non-Galilean spacetime transformation

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \frac{1}{2}\mathbf{g}t^2 \\ t' &= t. \end{aligned} \tag{23}$$

Then

$$m\frac{d^2\mathbf{x}'}{dt^2} = m\frac{d^2\mathbf{x}}{dt^2} - m\mathbf{g} = \mathbf{F}. \tag{24}$$

In other words, an experimenter who measures coordinates with respect to the box will finds that Newton's laws are obeyed, *but does not detect the gravitational field.*

This argument is fine if $\mathbf{g}$ is uniform and time-independent. If it is not uniform, then a *large* box can detect it, through the tidal forces which would, for example, draw together two particles in the Earth's gravitational field. Nevertheless, we can remove gravity to any required accuracy in a sufficiently small region and over a short enough period of time. This leads us to the **Weak Equivalence Principle** (WEP):

At any point in spacetime in an arbitrary gravitational field, it is possible to choose a freely-falling 'local inertial frame', in which the laws of motion are the same as if gravity were absent.

Note that there are *infinitely many* LIFs at any spacetime point, all related by Lorentz transformations. The WEP clearly only holds if $m_{\rm g} = m_{\rm i}$, and so it seems to amount just to a restatement of the unexplained fact that inertial and gravitational masses are exactly equal. But in 1907 Einstein took a radical further leap of intuition: if the gravitational field is undetectable in a local inertial frame, this is effectively saying that the field *does not exist* in this frame. He thus proposed the *Strong Equivalence Principle* (SEP):

In a local inertial frame, all SR laws of physics apply.

Hereafter, we shall assume this to be true and refer to it just as the 'EP'. It is a radical new perspective on gravity, which now appears almost as an illusion caused by viewing things in a frame that differs from the simple natural perspective of the freely-falling observer. The EP gives us one piece of solid ground to which we can always retreat: even in the strongest gravitational fields near a black hole, there is still a LIF, and we can assume we know all the laws of physics in that frame. One caveat is needed here, however. As we have seen, the gravitational acceleration ($-\boldsymbol{\nabla}\Phi$ in terms of the Newtonian potential) can always be transformed away – but this is not true for the tidal quantity $\partial^2\Phi/\partial x^i\partial x^j$, so this is the true gravitational field. The EP therefore implicitly smuggles in something we should state explicitly: the **principle of minimal gravitational coupling**. This states that the SR laws of physics as we deduce them in a laboratory on Earth include no explicit terms that depend on the magnitude of the gravitational tidal force. This cannot be guaranteed in advance, so it is partly a statement of preference for simple laws of physics; in the end, this has to be tested by experiment.

3.4 Gravitational time dilation

An an immediate illustration of the power of the EP, we can use it to learn an important new aspect of gravity, via the following thought experiment. Consider an accelerating frame, which is conventionally a rocket of height h, with a clock mounted on the roof that regularly disgorges photons towards the floor, as in Figure 1. If the rocket accelerates upwards at g, the floor acquires a speed $v = gh/c$ in the time taken for a photon to travel from roof to floor. There will thus be a blueshift in the frequency of received photons, given by $\Delta\nu/\nu = gh/c^2$, and it is easy to see that the rate of reception of photons will increase by the same factor.

Now, since the rocket can be kept accelerating for as long as we like, and since photons cannot be stockpiled anywhere, the conclusion of an observer on the floor of the rocket is that in a real sense the clock on the roof is running fast. When the rocket stops accelerating, the clock on the roof will have gained a time Δt by comparison with an identical clock kept on the floor. Finally, the equivalence principle can be brought in to conclude that gravity must cause the same effect.

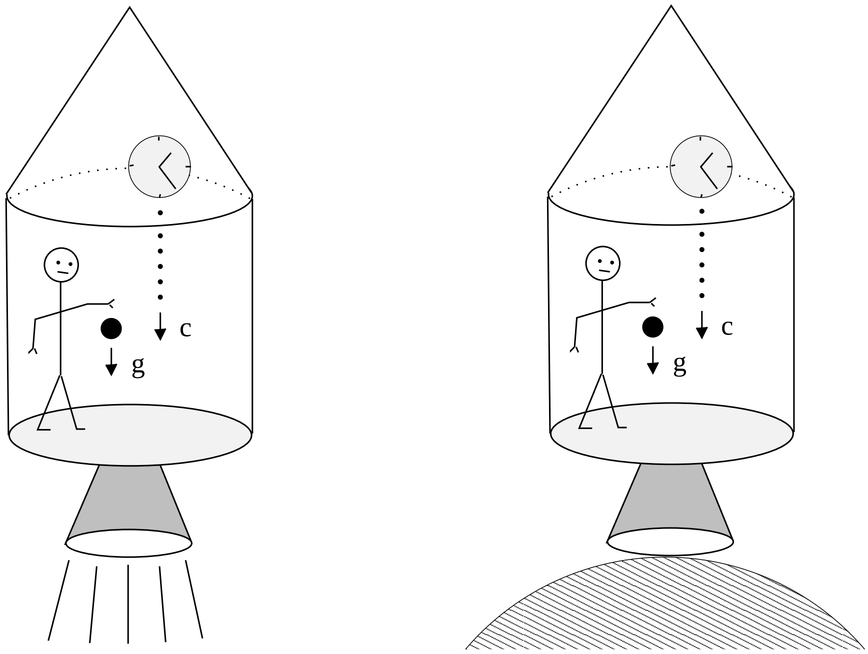

Figure 1: *Illustrating how an apparent gravitational field generates time dilation.*

Noting that $\Delta\Phi = gh$ is the difference in potential between roof and floor, it is simple to generalize this to

$$\boxed{\frac{\Delta t}{t} = \frac{\Delta\Phi}{c^2}}. \tag{25}$$

The same thought experiment can also be used to show that light must be deflected in a gravitational field: consider a ray that crosses the rocket cabin horizontally when stationary. This track will appear curved when the rocket accelerates. We will return to this point later.

The experimental demonstration of the gravitational redshift by Pound & Rebka (1960) was one of the main pieces of evidence for the essential correctness of the above reasoning, and provides a test (although not the most powerful one) of the equivalence principle.

A striking application of this concept is the resolution of the twin paradox in SR. The twin on a rocket experiences an accelerating frame of reference while it turns, so that clocks on the Earth undergo a gravitational speeding up, accounting exactly for the fact that the rocket-borne twin is younger on return to Earth. We will explore this calculation in the tutorials.

4 GR spacetime and equations of motion

We are now in a position to use the EP to obtain deep new information about the structure of spacetime and the GR equation of motion. We will do this by transforming motion from a locally flat (LIF) spacetime to an arbitrary coordinate system.

4.1 Exploiting the equivalence principle

Consider a freely-moving particle in a gravitational field. According to the EP, there is always a Local Inertial Frame (LIF) coordinate system $\xi^\alpha = (ct, \mathbf{x})$ in which the particle follows an unaccelerated trajectory. We can write write this trajectory **parametrically**, as a function, for example, of the proper time, $\xi^\alpha(\tau)$. The SR expression of zero acceleration is

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0. \tag{26}$$

To this, we should add the SR spacetime interval:

$$c^2 d\tau^2 = c^2 dt^2 - d\mathbf{x}^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta, \tag{27}$$

where the proper time interval, τ, is the time measured by a clock moving with the particle through spacetime. The spacetime of Special Relativity is called **Minkowski spacetime**. As before, $\eta_{\alpha\beta} \equiv \mathrm{diag}(1, -1, -1, -1)$. Here and throughout Greek indices on 4-vectors will run from 0 to 3, while spatial parts of the 4-vectors will be denoted by Latin indices that run from 1 to 3. The Einstein summation convention, where repeated indices are summed over, applies unless otherwise stated. Note that the repeated (dummy) indices normally have to be of opposite kinds (one upstairs, one downstairs): $A^\mu A_\mu$ is an invariant scalar, but $A^\mu A^\mu$ would not be. Where there is a non-dummy index, it should match on either side of the equation: $A^\alpha = B^\alpha$ just says that the 4-vectors A and B are the same, and it doesn't matter what the index is called.

Now consider *any* other arbitrary frame of reference, which may be accelerating or rotating, in which the particle coordinates are $x^\mu(\tau)$. Using the **chain rule** we can expand a small displacement in the local Minkowski spacetime in terms of the arbitrary coordinate system

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu. \tag{28}$$

A time derivative in the LIF can also be expanded in terms of the arbitrary coordinate system,

$$\frac{d}{d\tau} = \left(\frac{dx^\mu}{d\tau}\right)\frac{\partial}{\partial x^\mu}. \tag{29}$$

The SR equation of motion (26) becomes

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) &= 0, \\ \Rightarrow \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\right) &= 0, \\ \Rightarrow \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= 0. \end{aligned} \tag{30}$$

We see that the acceleration will be zero if the ξ^α are *linear* functions of the new coordinates x^μ, since $\partial^2\xi^\alpha/\partial x^\nu \partial x^\mu$ then vanishes. This is the case for Lorentz transformations. For a general transformation this term is non-zero. Note some subtleties of notation: there is a general transformation between coordinate systems ξ^α and x^μ, so the partial derivatives exist everywhere. The specific path of the particle is $\xi^\alpha(\tau)$ or $x^\alpha(\tau)$, thus the appearance of total derivatives of these quantities w.r.t. τ.

To find the acceleration in the new frame, multiply by $\partial x^\lambda/\partial \xi^\alpha$ and use the **product rule**, which follows directly from the chain rule (28):

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu, \tag{31}$$

where the right hand side is the **Kronecker delta** (=1 if $\lambda = \mu$ and zero otherwise). This gives us the equation of motion for a free particle, known as the **geodesic equation**:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0} \tag{32}$$

where ${\Gamma^\lambda}_{\mu\nu}$ is the **affine connection**:

$$ {\Gamma^\lambda}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}. \tag{33} $$

Note that Γ is symmetric in its lower indices, and, for future reference, it is *not* a tensor. The affine connection is sometimes written as $\left\{ {\lambda \atop \mu\nu} \right\}$ and called a **Christoffel symbol**.

The proper time interval can be written in terms of dx^μ through the line element:

$$ \begin{aligned} c^2 d\tau^2 &= \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} \left(\frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \right) \left(\frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu \right) \\ \Rightarrow c^2 d\tau^2 &= g_{\mu\nu}\, dx^\mu dx^\nu \end{aligned} \tag{34} $$

where $g_{\mu\nu}$ is the **metric tensor** for an arbitrary spacetime (note it is symmetric):

$$ \boxed{g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}.} \tag{35} $$

From this, we can see that the metric gives a generalisation of the way we defined covariant 4-vectors in Special Relativity. Now we adopt

$$ \boxed{V_\mu \equiv g_{\mu\nu}\, V^\nu,} \tag{36} $$

so that the squared norm of any 4-vector is $V_\mu V^\mu = g_{\mu\nu} V^\mu V^\nu$. Since 4-vectors are built out of dx^μ, the norm is invariant: this comes directly from the fact that $d\tau$ is invariant.

4.1.1 Massless particles

For massless particles, we cannot use $d\tau$, since it is zero. Instead we can use $\sigma \equiv \xi^0$ (ct in the LIF). Following similar logic, the condition $d^2\xi^\alpha / d\sigma^2 = 0$ becomes

$$ \frac{d^2 x^\lambda}{d\sigma^2} + {\Gamma^\lambda}_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0 \tag{37} $$

In neither this nor the massive particle case do we need to know what σ or τ are explicitly, since we have 4 equations to solve for e.g. $x^\mu(\tau)$, and τ can be eliminated to obtain the 3 equations $\mathbf{x}(t)$. Indeed, we see that the equation of motion will be the same replacing τ by any **affine parameter** that is linearly related to τ.

4.2 Physical implications

The above analysis is a straightforward change of variables applied to simple equations in a LIF, but the implications for *physics* are profound. We have learned two important things:

(1) Gravitational forces are velocity dependent.

(2) Spacetime is (probably) curved.

The first of these statements follows by noting that the 4-acceleration in the transformed frame is quadratic in the 4-velocity, so there will be at least terms linear in the velocity. This is encouraging, as it makes gravity more like electromagnetism, where the force on a charged particle is the **Lorentz force**, $\mathbf{v} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$. The analogy between Newtonian gravity and electrostatics works very well, and so we should expect that a relativistic theory of gravity would extend this analogy so that we also have **gravomagnetic fields** generated by the motion of mass. We also expect that there should be gravitational waves, so that gravitational effects propagate at the speed of light, rather than being an instantaneous action-at-a-distance. As we will see, this expectation is correct.

But the more radical item is the second one. Using the EP, we have shown that spacetime must have a **metric structure** $c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, where the **metric tensor** is some matrix that differs from the simple constant **Minkowski metric**, $\eta_{\mu\nu}$. The existence of a non-trivial metric is a big step towards showing that spacetime is curved. Think of a simple example like the element of length on the surface of a unit sphere in spherical polar coordinates, (θ, ϕ): this is $d\ell^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The idea that spacetime 'lengths' may have to be described with a similarly complicated metric is a big hint that we should be thinking about spacetime curvature. The hint falls short of a proof, however, as one can always rewrite something that lacks curvature using a more complicated coordinate system. For example, lengths on a flat 2D plane can be written using (r, θ) polars as $d\ell^2 = dr^2 + r^2 d\theta^2$, but the appearance of this more complicated metric doesn't mean that the plane has suddenly become curved. We need a way to describe spacetime curvature in a way that is independent of coordinates, and this will be dealt with later in the course.

But again things seem to be going in a desirable direction, as Einstein had an expectation that GR would involve spacetime curvature right from the start. He was heavily influenced by the example of the rotating disc. Consider a stationary disc at radius r, where a transverse element of length is $d\ell = r\, d\theta$. Integrating round in θ, we learn that the circumference is $2\pi r$. But now set up a rotating disc, containing observers with metre rulers that they set down tangentially. As we in the rest frame observe them, each ruler is length contracted by a factor γ (the Lorentz factor based on the rotational velocity at a given r) – so now we need more of these rulers to fit round the circumference. Hence an observer living in the disc will conclude that its circumference is $\gamma 2\pi r$, so that the geometry in the rotating frame is non-Euclidean.

4.3 The metric as the gravitational field

At present, both the metric and the affine connection are expressed in terms of some unknown coordinate transformation. But we now show that this transform can be eliminated, so that the affine connection and hence particle dynamics is determined once the metric is given. From (35),

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\,\eta_{\alpha\beta}, \tag{38}$$

we have

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\,\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\,\eta_{\alpha\beta}, \tag{39}$$

and from the definition of the affine connection (33), we see that

$$\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} = \Gamma^\rho{}_{\mu\nu}\frac{\partial \xi^\alpha}{\partial x^\rho}. \tag{40}$$

So (39) becomes

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho{}_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\,\eta_{\alpha\beta} + \Gamma^\rho{}_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\,\eta_{\alpha\beta}. \tag{41}$$

Using (35), this simplifies to

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho}. \tag{42}$$

Now relabel indices: first $\mu \leftrightarrow \lambda$:

$$\frac{\partial g_{\lambda\nu}}{\partial x^\mu} = \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho{}_{\mu\nu} g_{\lambda\rho}. \tag{43}$$

Second: $\nu \leftrightarrow \lambda$:

$$\frac{\partial g_{\mu\lambda}}{\partial x^\nu} = \Gamma^\rho{}_{\nu\mu} g_{\rho\lambda} + \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho}. \tag{44}$$

Add the first two of these, and subtract the last, and use the symmetry of Γ w.r.t. its lower indices, to get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\Gamma^\rho{}_{\lambda\mu} g_{\rho\nu}. \tag{45}$$

Now we *define* the inverse of the metric tensor as $g^{\sigma\rho}$, by

$$\boxed{g^{\sigma\rho} g_{\rho\nu} \equiv \delta^\sigma{}_\nu\ ,} \tag{46}$$

where $\delta^\sigma{}_\nu$ is the unit matrix: 1 if $\nu = \sigma$ and 0 otherwise. Note that $g^{\sigma\rho}$ and $g_{\sigma\rho}$ are both symmetric. Hence

$$\boxed{\Gamma^\sigma{}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right).} \tag{47}$$

We thus see that the gravitational term in the geodesic equation depends on the *gradients* of $g_{\mu\nu}$, justifying the description of the metric components as gravitational **potentials** (cf. $\mathbf{g} = -\nabla\Phi$ in Newtonian gravity). Note that there are 10 potentials, instead of one in Newtonian physics (why 10 and not 16?). If we have $g_{\mu\nu}(x^\alpha)$, we can then solve the geodesic equation and determine the orbit. In general, this is the way to proceed, but if the problem has some symmetry to it, then a variational approach is easier, as explained below.

Thus everything in gravitational dynamics derives from the metric: but where does the metric come from? We will postpone the answer to this question for a while, but the ultimate answer is that $g_{\mu\nu}$ is obtained from the solution of Einstein's field equations, which are the relativistic generalisation of Poisson's equation for the Newtonian potential, $\nabla^2\Phi = 4\pi G\rho$.

4.4 Newtonian limit of the geodesic equation

If speeds are $\ll c$, and the gravitational field is weak and stationary, then the geodesic equation (32),

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \tag{48}$$

can be approximated by ignoring the $d\mathbf{x}/d\tau$ terms in comparison with $d(ct)/d\tau$. Then

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{00} c^2 \left(\frac{dt}{d\tau} \right)^2 \simeq 0. \tag{49}$$

For a stationary field, $\partial g_{\mu\nu}/\partial t = 0$, so the affine connection (33) is

$$\Gamma^\lambda{}_{00} = \frac{1}{2} g^{\nu\lambda} \left(\frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu} \right) = -\frac{1}{2} g^{\nu\lambda} \frac{\partial g_{00}}{\partial x^\nu}. \tag{50}$$

For a *weak field*, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{51}$$

and assume $|h_{\mu\nu}| \ll 1$. To first order in h,

$$\Gamma^\lambda{}_{00} = -\frac{1}{2} \eta^{\nu\lambda} \frac{\partial h_{00}}{\partial x^\nu}, \tag{52}$$

so *gravitational forces in the Newtonian limit are determined entirely by gradients in* g_{00}. For a stationary field the above sum only involves the spatial indices ($\nu = 1, 2, 3$) of η, which have the value -1 just along the diagonal ($\nu = \lambda$). Thus

$$\eta^{\nu\lambda} \frac{\partial h_{00}}{\partial x^{\nu}} = -\frac{\partial h_{00}}{\partial x^{\lambda}}. \tag{53}$$

The geodesic equation (49) then becomes

$$\frac{d^2 x^{\lambda}}{d\tau^2} = -\frac{1}{2} c^2 \left(\frac{dt}{d\tau}\right)^2 \frac{\partial h_{00}}{\partial x^{\lambda}}, \tag{54}$$

with spatial parts

$$\frac{d^2 \mathbf{x}}{d\tau^2} = -\frac{1}{2} c^2 \left(\frac{dt}{d\tau}\right)^2 \nabla h_{00} \tag{55}$$

From the $\lambda = 0$ equation, we find ${\Gamma^0}_{00} = 0$. The equation of motion is then $d^2 x^0/d\tau^2 = 0$ which has solution $dt = A d\tau$, where A is some constant. We can substitute $d\tau$ for dt/A in equation (55) and cancel the factors of A on both sides, and using $dt/dt = 1$, we find

$$\frac{d^2 \mathbf{x}}{dt^2} = -\frac{1}{2} c^2 \nabla h_{00}. \tag{56}$$

Comparing with the Newtonian result $d^2\mathbf{x}/dt^2 = -\nabla\Phi$, we conclude that

$$h_{00} = \frac{2\Phi}{c^2}, \tag{57}$$

plus a constant, which we take to be zero if we follow the convention that $\Phi \to 0$ far from any masses (where the metric approaches that of SR and $h \to 0$). Hence, in the *weak-field limit*,

$$\boxed{g_{00} = 1 + \frac{2\Phi}{c^2}}, \tag{58}$$

which as we saw above is the only one of the $g_{\mu\nu}$ 'potentials' that contributes in Newtonian gravity. But in general gradients in all of the $g_{\mu\nu}$ contribute to the affine connection and thus to the effective gravitational force in a more general situation. Thus for stronger gravitational fields the effects of spatial curvature may also become important.

4.5 Gravitational time dilation and redshift

The fact that g_{00} is not unity relates to our earlier discussion of gravitational time dilation. Consider a moving particle that carries a clock: as before, the **proper time** is the time elapsed on this clock in the rest frame of the particle. We know that this time is given by the metric relation

$$c^2 d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{59}$$

so for a *stationary* clock (whose spatial coordinates are fixed, so $dx^i = 0,\ i = 1, 2, 3$),

$$c^2 d\tau^2 = g_{00}\, c^2 dt^2 \tag{60}$$

so that $d\tau = \sqrt{g_{00}}\, dt$. For a clock in a gravitational field, in the weak-field case, where $g_{00} = 1 + 2\Phi/c^2$,

$$d\tau \simeq \left(1 + \frac{\Phi}{c^2}\right) dt \tag{61}$$

and we see that t coincides with τ only if $\Phi = 0$.

Now, here is a subtle point. Although we are considering events taking place inside a gravitational potential well, $x^\mu = (ct, x, y, z)$ is presumed to be a global coordinate system – so the spacetime interval between a pair of events at one location can be agreed on by observers at all locations (in principle: in practice measuring such intervals would require allowance for the propagation of light signals). Thus we can say that *dt is the time elapsed on a stationary clock at infinity*, where $\Phi = 0$. We see that, from the point of view of this observer, the stationary clock deep inside the potential well runs slow, by exactly the amount we deduced using the equivalence principle.

In our direct argument from the equivalence principle, we showed that time dilation was accompanied by gravitational redshifting of the frequency of received photons. We can now derive this effect more formally, as follows. Consider a stationary light emitter at position $\mathbf{x}_1$ and a stationary observer at $\mathbf{x}_2$. Let the emitted EM field be at a maximum at time t_1 and again at $t_1 + dt_1$ (i.e. dt_1 is the period of the emitted radiation. The EM signal propagates at the speed of light and these two peaks arrive at times t_2 and $t_2 + dt_2$. Now, radiation must follow a **null trajectory** with $d\tau = 0$. If the metric is time independent, then the line element can be integrated to find the time interval: $t_2 - t_1 = \int f(x)\, dx$, where $f(x)$ is some spatial function derived from the metric. Now, because the metric is time independent, this coordinate time interval must be the same for all journeys, and so we learn that $dt_1 = dt_2 = dt$. We can write this in terms of the proper time intervals measured at the two points as

$$\frac{d\tau_1}{\sqrt{g_{00}(\mathbf{x}_1)}} = dt = \frac{d\tau_2}{\sqrt{g_{00}(\mathbf{x}_2)}} \Rightarrow \frac{d\tau_1}{d\tau_2} = \sqrt{\frac{g_{00}(\mathbf{x}_1)}{g_{00}(\mathbf{x}_2)}}. \tag{62}$$

This is a ratio of emitted and observed periods, so it is the ratio of observed and emitted frequencies. The factor $\sqrt{g_{00}}$ is the same one that dilates the apparent ticking of clocks, as expected. In weak gravitational fields, $g_{00} \simeq 1 + 2\Phi/c^2$ (equation 58), so to $O(\Phi)$ we have

$$\frac{\nu_2}{\nu_1} \simeq \sqrt{\frac{(1 + \frac{2\Phi_1}{c^2})}{(1 + \frac{2\Phi_2}{c^2})}} \simeq 1 + \frac{\Phi_1}{c^2} - \frac{\Phi_2}{c^2} \tag{63}$$

and the **gravitational redshift** , defined here as $1 + z_{\rm grav} = \nu_1/\nu_2$, is

$$\boxed{z_{\rm grav} = \frac{\nu_1}{\nu_2} - 1 \approx \frac{\Phi_2 - \Phi_1}{c^2}.} \tag{64}$$

Since Φ is negative near massive bodies, a photon will then lose energy (be redshifted) by climbing out of a gravitational well (or gain energy and be blueshifted when travelling into one). This is small for most astronomical bodies ($\sim 10^{-6}$ for the Sun), and often masked by Doppler effects e.g. convection in Sun, which gives systematic effects that are larger than this.

5 Variational formulation of GR

Dynamical equations in the form of differential equations may be written as a variational principle. This approach is used extensively in classical mechanics and can readily be applied in General Relativity. This approach offers some powerful advantages in calculations.

5.1 Stationary intervals

A particle follows a worldline between spacetime points A and B. Let p be a parameter that increases monotonically along the worldline, so that the proper time elapsed is

$$c\,\tau_{AB} = c\int_A^B d\tau = c\int_A^B \frac{d\tau}{dp}\,dp = \int_A^B L(x^\mu, \dot{x}^\mu)\,dp. \tag{65}$$

Here we have written the integral in a way that makes it look like the **action integral** of Lagrangian mechanics, $\int L\,dt$, where $L = T - V$ is the difference of kinetic and potential energies. When the starting and ending points A & B are fixed, classical mechanics in the Lagrangian formalism is recovered if we require that the particle trajectory is **stationary**, i.e. unchanged when we make small perturbations in the particle trajectory $\mathbf{x}(t)$. Variational calculus then requires the **Euler-Lagrange equation** for each degree of freedom, q:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}\right) = 0. \tag{66}$$

For a single particle, the degrees of freedom are the spatial positions x^i: with parameter $p = t$ and $L = m|\dot{\mathbf{x}}|^2/2 - V$, we get $m\ddot{x}^i = -\partial V/\partial x^i$, as required.

In SR, it is easy to show that something similar is going on, and that the integral for the proper time interval is stationary (actually a maximum). Here, the equivalent of the Lagrangian is

$$L = c\frac{d\tau}{dp} = \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}} = \left[(c\dot{t})^2 - (\dot{x})^2 - (\dot{y})^2 - (\dot{z})^2\right]^{1/2}, \tag{67}$$

where $\dot{t} \equiv dt/dp$ etc. – note we will frequently find it convenient to use dots to denote parameter derivatives in this way. For a free particle, we would expect the Euler-Lagrange equation based on this L to give the trajectory $x^i = A + Bt$, i.e. linear motion. In tutorial sheet 2, you are encouraged to prove this working from the above expression for L. But the square root makes this slightly awkward, and there is an simpler way, which is to replace L by L^2 in the Euler-Lagrange equation. The justification for this is easy, if a little odd: since $c\Delta\tau = \int c\,d\tau$, then $L = c$ if we use τ as the parameter p, in which case $L^2 = c^2$. Then we have $\partial(L^2)/\partial x^\mu = 0$, and so the Euler-Lagrange equation can be integrated immediately to get $x^\mu = A + B\tau$, where A and B are different coefficients for each coordinate – so all are linearly related to each other and hence the particle moves in a straight line at constant velocity. Any accelerated trajectory that deviates from this will thus have a shorter proper time. This is the SR 'solution' of the twin paradox, although it is actually an evasion, since it refuses to analyse things from the point of view of the accelerated observer.

By the Equivalence Principle, and because τ is an invariant, we would expect the idea of stationary proper time to be maintained – i.e. particles should in general travel on **geodesics**: stationary trajectories in spacetime. It is immediately obvious how to generalise the SR analysis, since we have seen that the spacetime interval involves the general metric: $c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$. Therefore the geodesic principle is

$$\delta\int L\,dp = 0; \quad L^2 = g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}. \tag{68}$$

As before, we argue that it is possible (and more convenient) to use the Euler-Lagrange equation with L^2 instead of L. We can make the justification of this a little more formal. Consider the Euler-Lagrange equation for L:

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0, \tag{69}$$

where again $\dot{x}^\mu \equiv dx^\mu/dp$. Now consider the same equation for L^2:

$$\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = -2\frac{dL}{dp}\frac{\partial L}{\partial \dot{x}^\mu}. \tag{70}$$

We can make the RHS zero by noting that since $L = c d\tau/dp$, we have $dL/dp = c d^2\tau/dp^2$, which can be made to vanish *if* we choose p to be an **affine parameter**, which is any parameter that increases *linearly* with τ. You might wonder why we are going to all the trouble of considering a parameter p that differs from τ in this simple way, and the reason is photons. Since massless particles obey the null condition $\tau = 0$, we need a different parameter to distinguish points on their trajectories, as we saw earlier in discussing the geodesic equation.

So in summary we expect free particles to travel along geodesics in spacetime. Given an affine parameter p, the trajectory should obey the L^2 form of the Euler-Lagrange equation:

$$\boxed{\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = 0. \qquad \text{ELII}} \tag{71}$$

We will use this 'ELII' equation extensively for computations, as it is often more practically convenient than considering the geodesic equation and evaluating the affine connection directly from equation (33). It can in effect be regarded as a short-cut for computing these coefficients. For completeness, we should really check that the statement that τ_{AB} is stationary is indeed equivalent to the geodesic equation (32). We have taken this result as obvious through the equivalence principle; to prove it directly is a slightly messy exercise, and it is given as a problem in Tutorial 2.

5.2 Calculating the affine connection

We now illustrate the use of the variational route to obtaining the affine connection. In detail, the procedure is:

- Write down the relativistic line element for the spacetime.
- Convert this to L^2 ($dx^\mu \to \dot{x}^\mu$).
- Write down the ELII equation for each variable (recall x^μ and $\dot{x}^\mu$ are independent variables).
- Rearrange ELII to get it in the form: $\ddot{x}^\lambda + [\text{ something }]^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$.
- Read off the affine connection terms, $\Gamma^\lambda{}_{\mu\nu} = [\text{ something }]^\lambda_{\mu\nu}$ (ensure to match the indices).
- If $\mu \neq \nu$, divide by 2. $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ and μ and ν are summed over in the geodesic equation, but your expression probably won't have a sum.
- Any Γ not appearing is zero.

Example: A geodesic arc on the 2D surface of a sphere of radius R (this example does not involve time, but the approach is general). The line element between points with coordinates θ, ϕ separated by $d\theta, d\phi$ is

$$\begin{aligned} d\ell^2 &= R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2 \\ \Rightarrow L^2 &= R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2. \end{aligned} \tag{72}$$

Apply the ELII equations, first to θ:

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(R^2\dot{\theta}^2+R^2\sin^2\theta\,\dot{\phi}^2\right)-\frac{d}{dp}\left[\frac{\partial}{\partial\dot{\theta}}\left(R^2\dot{\theta}^2+R^2\sin^2\theta\,\dot{\phi}^2\right)\right] &= 0\\ \Rightarrow 2R^2\sin\theta\cos\theta\dot{\phi}^2-2R^2\ddot{\theta} &= 0.\end{aligned}\tag{73}$$

Hence θ has an 'equation of motion' (but no motion, in this case)

$$\ddot{\theta}-\sin\theta\cos\theta\,\dot{\phi}^2=0.\tag{74}$$

We can simply read off the affine connection, noting the $\dot{\phi}$ factors tell us the lower indices,

$$\Gamma^{\theta}{}_{\phi\phi}=-\sin\theta\cos\theta.\tag{75}$$

Now, for ϕ:

$$\begin{aligned}\frac{\partial}{\partial\phi}\left(R^2\dot{\theta}^2+R^2\sin^2\theta\,\dot{\phi}^2\right)-\frac{d}{dp}\left[\frac{\partial}{\partial\dot{\phi}}\left(R^2\dot{\theta}^2+R^2\sin^2\theta\,\dot{\phi}^2\right)\right] &= 0\\ \Rightarrow 2R^2\left(2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi}+\sin^2\theta\,\ddot{\phi}\right) &= 0,\end{aligned}\tag{76}$$

and so the equation of motion in the ϕ-direction is

$$\ddot{\phi}+2\cot\theta\,\dot{\theta}\dot{\phi}=0.\tag{77}$$

Remembering to divide by 2, we can again read off the affine connection terms, $\Gamma^{\phi}{}_{\phi\theta}=\Gamma^{\phi}{}_{\theta\phi}=\cot\theta$. Equations (74) and (77) are the equations describing a geodesic arc on the sphere, i.e. a great circle. The non-trivial form (deviation from a linear relation of the coordinates) arises from the curvature of the space. It will be convenient for later purposes to write the affine connection as two 2×2 matrices $(\Gamma_\theta)^{\alpha}{}_{\beta}\equiv\Gamma^{\alpha}{}_{\theta\beta}$:

$$\Gamma_\theta=\begin{pmatrix}\cdot & \cdot\\ \cdot & \cot\theta\end{pmatrix}\qquad\Gamma_\phi=\begin{pmatrix}\cdot & -\sin\theta\cos\theta\\ \cot\theta & \cdot\end{pmatrix}.\tag{78}$$

6 Example application: the Schwarzschild metric

We now illustrate the application of the tools acquired so far to a specific interesting case – leaving until later the critical question of how we know what the metric is in a given situation. The Schwarzschild metric is the GR gravitational field of a point mass in empty space. It is commonly used as a model for a static Black Hole, or for the metric outside of a star or neutron star where the gravitational field can be large. The Schwarzschild metric has a greater significance in GR, since **Birkhoff's Theorem** shows that it describes the spacetime outside any general spherically symmetric mass distribution. This is interesting because this includes distributions that have any radial profile and which are time-dependent. This last feature implies that a time-dependent spherical system cannot induce a time-dependency in the surrounding spacetime. More of this later.

6.1 Distances on the 2-sphere

Before we delve into the Schwarzschild spacetime, it is worth pausing for a moment to consider that the coordinate freedom implied by General Relativity allows us to express the same spacetime in different ways. This freedom allows us some choice in the coordinate system, so we can select coordinates that simplify the representation of a spacetime. We shall make use of this coordinate freedom in our choice of coordinates in the Schwarzschild spacetime.

As a simple example, let us again consider the 2D surface of a sphere. Previously we used spherical angles θ and ϕ to label the surface, giving equation (72). If we let $r \equiv R\theta$ (r is called the **geodesic distance**), then

$$d\ell^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi^2. \tag{79}$$

Here the first term is as simple as we can make it, but the second is complicated (except when $R \to \infty$, when we get the flat plane in polar coordinates $dr^2 + r^2 d\phi^2$). But there are infinitely many alternatives. For example, we can try to make the coefficient of $d\phi^2$ as simple as possible, by taking $\rho \equiv R\sin(r/R)$, so $d\rho = \cos(r/R)\, dr = \sqrt{1-\rho^2/R^2}\, dr$, and

$$\boxed{d\ell^2 = \frac{d\rho^2}{1-\kappa\rho^2} + \rho^2\, d\phi^2,} \tag{80}$$

where $\kappa \equiv 1/R^2$ is the curvature of the sphere. In this case the radial term is complicated by the curvature of the surface, while the angular part now looks like a flat plane. In these coordinates the radial coordinate ρ is called the **angular diameter distance**, defined so that the angle subtended by a rod of length $d\ell_\perp$ perpendicular to the line-of-sight is $d\phi = d\ell_\perp/\rho$.

Note that equation (80) applies to spheres ($\kappa > 0$), flat surfaces ($\kappa = 0$), but also to **hyperbolic** negatively-curved surfaces ($\kappa < 0$), which are less easy to visualise in 3D. Horse saddles have negative curvature at a point, and it is possible to construct surfaces such as the trumpet-like **pseudosphere** that have constant negative curvature – although these are not unbounded in the same way as the positive-curvature sphere. A distinctive feature of the sign of the curvature is the effect on a triangle. On a flat surface the interior angles of a triangle add up to 180°. On a sphere the interior angles are > 180°, while on a negatively curved surface they add up to < 180°.

6.2 SR metric in spherical coordinates

The line element in Special Relativity is $c^2 d\tau^2 = c^2 dt^2 - d\ell^2$, where the spatial part of the line element $d\ell^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates, or $d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ in spherical coordinates (r, θ, ϕ). It is instructive to note that

$$d\ell^2 = dr^2 + r^2 d\psi^2, \tag{81}$$

where $d\psi^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the square of the angle between radial lines separated by $(d\theta, d\phi)$. The SR metric may therefore be written

$$c^2 d\tau^2 = c^2 dt^2 - \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{82}$$

6.3 Schwarzschild metric in spherical coordinates

The Schwarzschild metric describes the empty but curved spacetime around a general spherically-symmetric mass distribution, for example around a point mass. The spherical symmetry of the situation specifies much of the form of the metric. Firstly, the symmetry suggests the use of spherical polars. For r we choose to use the angular diameter distance, so the perpendicular part of the metric is $r^2 d\psi^2$ (this *defines* r). Because spacetime is curved, we do not expect to see Minkowski spacetime terms like dr^2 or c^2dt^2 (remember we expect g_{00} to be modified by the spatially-dependent gravitational potential in weak fields), but rather

$$c^2d\tau^2 = e(r)c^2dt^2 - \left[f(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{83}$$

By isotropy, e and f cannot depend on direction. If we assume that the metric is *stationary*, then they won't depend on t either (as mentioned earlier, the metric can be stationary even if the mass distribution is not). At large distances from the point mass we will impose the physically sensible boundary condition that the metric tends to SR, so $e, f \to 1$ as $r \to \infty$. A dimensional analysis also implies that e and f depend on $GM/(rc^2)$, and we already know that in the weak field limit, $g_{00} \simeq 1 + 2\Phi/c^2$.

This is as far as we can go without resorting to Einstein's field equations to obtain the exact solution, and for now we simply quote the result:

$$\boxed{c^2d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).} \tag{84}$$

We note that:

- t is **coordinate time**, corresponding to time measured by *stationary clocks at* ∞. The **proper time** elapsed, for a stationary clock at (r, θ, ϕ) is $d\tau = dt\sqrt{1 - 2GM/rc^2}$.
- The coefficient of dt^2 agrees with our weak-field calculation when $r \gg GM/(c^2)$,
- Not only is time curved, with $g_{00} = 1 + 2\Phi/c^2$, but the spatial curvature has a similar dependence on Φ to linear order: $-g_{rr} \simeq 1 - 2\Phi/c^2$.
- Something strange happens at $r = 2GM/c^2 \equiv r_S$, the **Schwarzschild radius**. Putting $d\tau = 0$, we see that the coordinate speed of light, dr/dt, is zero at $r = r_S$. So there is an **event horizon** and light signals apparently cannot reach $r < r_S$. More on this later.

7 Orbits in the Schwarzschild metric

We are now in a position to derive the equations of motion for a test particle moving in the Schwarzschild metric, and study its possible orbits. We start by applying the Euler-Lagrange equations (71) to the Schwarzschild metric:

$$\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = 0. \tag{85}$$

As before for matter particles we take $p = \tau$, for which $L^2 = c^2(d\tau/dp)^2 = c^2$ along the correct path. In general, L^2 is

$$L^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{rc^2}} - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right), \tag{86}$$

where the dot indicates now $d/d\tau$. If we write $\alpha \equiv 1 - 2GM/(rc^2)$, then the ELII equations for variables t, θ, and ϕ are respectively

$$\begin{aligned} -\frac{d}{dp}\left(2c^2\alpha\dot{t}\right) &= 0 \\ 2r^2\sin\theta\cos\theta\,\dot{\phi}^2 - \frac{d}{dp}\left(-2r^2\dot{\theta}\right) &= 0 \\ -\frac{d}{dp}\left(-2r^2\sin^2\theta\,\dot{\phi}\right) &= 0. \end{aligned} \tag{87}$$

The time equation of motion gives

$$\boxed{\alpha \dot{t} = \text{constant} = k,} \tag{88}$$

which says that there is time dilation. We can see that the form makes sense by considering non-relativistic weak fields. We would expect Doppler and gravitational time dilation, so $\dot{t} = \gamma(1 + GM/c^2 r) \simeq 1 + v^2/2c^2 + GM/c^2 r$. Since $v^2/2 - GM/r = E$, the total energy per unit mass, we get something equal to the GR equation to first order, with $k = 1 + E/c^2$ – or just $k = E/c^2$, if we include the rest mass in the total energy.

Without loss of generality we can define the orbit to lie in the equatorial plane, $\theta = \pi/2$ and $\dot{\theta} = 0$; this choice satisfies the 2nd geodesic equation, and is intuitively reasonable given the spherical symmetry of the metric. With this choice, the third ELII equation gives

$$\boxed{r^2 \dot{\phi} = \text{constant} = h,} \tag{89}$$

which clearly expresses conservation of angular momentum.

The radial coordinate, r, in the ELII equation will yield the radial acceleration equation. However, a more useful form is the radial 'energy' equation, which we can derive directly from the Lagrangian-squared. We use the fact that $L^2 = c^2$ for a massive particle, so

$$\begin{aligned} c^2 &= c^2 \alpha \frac{k^2}{\alpha^2} - \frac{\dot{r}^2}{\alpha} - r^2 \frac{h^2}{r^4}, \\ \Rightarrow \dot{r}^2 + \alpha \frac{h^2}{r^2} &= c^2(k^2 - \alpha) = c^2 k^2 - c^2 + \frac{2GM}{r}. \end{aligned} \tag{90}$$

Compare this with Newtonian orbits,

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} + \frac{v_\perp^2}{r} = -\frac{GM}{r^2} + \frac{h_{\rm N}^2}{r^3}, \tag{91}$$

where $h_{\rm N} = v_\perp r$ is the Newtonian specific angular momentum. Multiplying by dr/dt and integrating gives

$$\boxed{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{h_{\rm N}^2}{2r^2} - \frac{GM}{r} = \text{constant}.} \tag{92}$$

We can compare this Newtonian radial equation with the GR result (90) which can be cast as

$$\boxed{\frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} - \frac{GM}{r} - \frac{GMh^2}{r^3 c^2} = c^2(k^2 - 1)/2 = \text{constant}.} \tag{93}$$

So we see the form of these equations is the same (but note the different formal definitions of t, τ, r and h in the Newtonian and GR cases), but there is an extra term in the GR equations that couples the gravitational field to the angular momentum. This has the same sign as the gravitational potential and so is an extra attractive radial force. If we trace the origin of this extra force, it arises from the α factor in the radial term of the line element, i.e. the fact that the Schwarzschild metric contains spatial curvature in addition to the $g_{00} \neq 1$ that is required by the Equivalence Principle.

7.1 Effective potentials

We have arrived at a set of equations of motion for the orbits of massive particles in a Schwarzschild spacetime that look similar to those of Newtonian gravity, but with a more complicated potential.

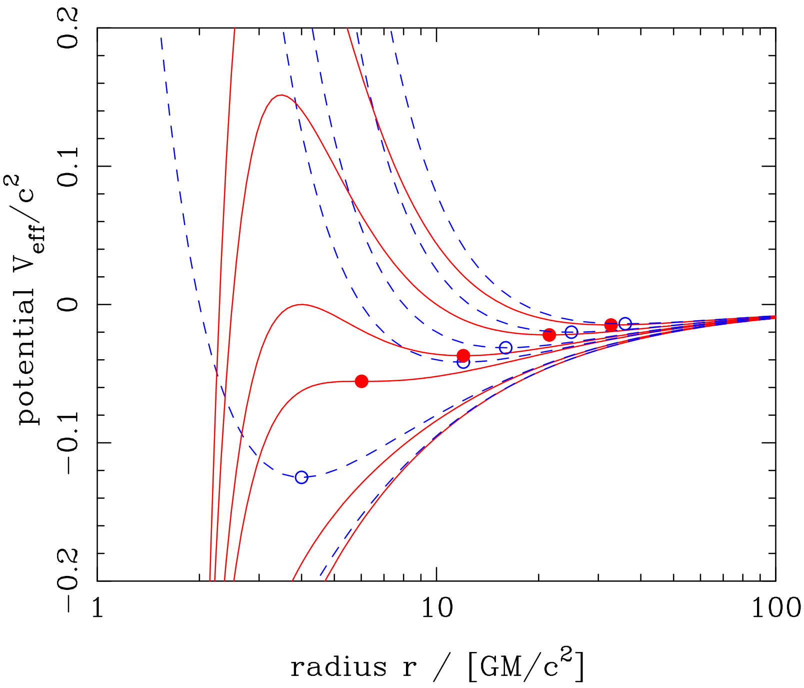

Figure 2: The effective GR potential around a point mass, and its Newtonian counterpart (dashed lines). These are plotted for different values of the dimensionless angular momentum, $H = r^2\dot{\phi}c/GM$. From bottom to top, the lines are $H = 1, 2, \sqrt{12}, 4, 5, 6$. Circular orbits live at minima in $V_{\rm eff}$, and are plotted as points (open for Newtonian). For $H < \sqrt{12}$, $V_{\rm eff}$ has no minimum, and so there is an innermost stable orbit, at $r = 6GM/c^2$.

We can therefore find the solutions to orbits using the intuitively familiar apparatus of potential fields. But the additional potential means that the solutions will be more complicated than the Newtonian ones. We have

$$\frac{\dot{r}^2}{2} + V_{\rm eff} = \text{constant}, \tag{94}$$

where

$$V_{\rm eff} = -\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{r^3c^2} \tag{95}$$

We can make this more appealingly dimensionless if we define a 'gravitational radius' $r_g \equiv GM/c^2$ (so the Schwarzschild radius $r_{\rm S} = 2r_g$), giving a dimensionless radius $R \equiv r/r_g$ and a dimensionless angular momentum $H \equiv h/r_g c$:

$$\frac{V_{\rm eff}}{c^2} = -\frac{1}{R} + \frac{H^2}{2R^2} - \frac{H^2}{R^3}. \tag{96}$$

As we saw earlier, the last term is new, and adds an attractive potential at small r, proportional to the squared angular momentum. As a result orbits now depend *qualitatively* on the value of H.

The equation for the 'radial kinetic energy' with an effective potential is highly informative. The simplest orbits will be circular ones with $\dot{r} = 0$, and these will lie at minima of the effective potential. For the Newtonian case, there is always a minimum: as we come in from ∞, the potential becomes progressively more negative; but eventually the **centrifugal barrier** $H^2/2R^2$ will come to dominate, so that a particle can never reach the origin. For these circular orbits, the 'energy' from the sum of $\dot{r}^2/2$ and $V_{\rm eff}$ is $V_{\rm min}$ – and this is also true in GR. If we raise the energy above this minimum, which keeping H fixed, there will be radial oscillations, leading to an elliptical orbit in the Newtonian case but (as we shall see) something more complicated in GR.

For a better understanding of circular orbits, it is helpful to differentiate the radial energy equation to get a force law. Since $d/dr = (\dot{r})^{-1} d/d\tau$, we get $\ddot{r} = -dV_{\rm eff}/dr$, so the condition for a circular orbit is not just $\dot{r} = 0$: we also need the gradient of $V_{\rm eff}$ to vanish. Solving for $dV_{\rm eff}/dr = 0$ gives $H = R/\sqrt{R-3}$, or $R = (H^2 \pm \sqrt{H^4 - 12H^2})/2$, as compared with $R = H^2$ in the Newtonian case. This looks puzzling: a given R requires a fixed H, and for a given H, there are two possible values of R. We can see what is going on by differentiating again, and we find that the two stationary points for given H have opposite 2nd derivatives: the outer one is a minimum and hence is stable. The inner one is a maximum, and hence circular orbits there would not be stable. For $H = \sqrt{12}$, these two merge, and for smaller H there is no minimum in $V_{\rm eff}$. Hence there is an **innermost stable orbit** at this point: $R = 6$. Anyone trying to orbit inside this radius risks an instability that would take them spiralling in to $r = 0$. This is illustrated in Figure 2, where we can also note that for $H = 4$ there is a **marginally bound orbit** at $R = 4$, where $V_{\rm eff} = 0$ at the orbit. Finally, the extreme limit of the unstable orbits at high H is $R = 3$: within this radius, circular orbits are not possible.

The significance of the innermost orbit at $R = 3$ may be guessed from a comparison with the Newtonian case. Newtonian orbits have a velocity $v = L/r$, where L is the angular momentum per unit mass. Since we have seen that $L \propto \sqrt{r}$, so the Newtonian orbital velocity increases as r becomes smaller. Hence we may guess that the limit at $R = 3$ corresponds to the orbital velocity reaching c – i.e. $R = 3$ corresponds to the **photon sphere** at which light would orbit a black hole. We can verify this intuition as follows. The geodesic equations for photons are almost the same as the ones for massive particles, except that the affine parameter now has to be something other than τ, which is unchanging. With this change, the first two Euler-Lagrange equations are identical: $\alpha \dot{t} = k$ and $r^2 \dot{\phi} = h$. But for photons the null interval means we replace $L^2 = c^2$ for a massive particle by $L^2 = 0$. Thus the radial energy equation becomes

$$\frac{\dot{r}^2}{2} = \frac{c^2 k^2}{2} - \frac{h^2 \alpha}{2r^2} = 0. \tag{97}$$

So now the effective potential is much simpler: $V_{\rm eff} = h^2\alpha/2r^2$, with the Newtonian $-GM/r$ term entirely absent. The condition for a circular orbit is that this has zero gradient, which is easily seen to require $R = 3$. But even though $V_{\rm eff}$ has changed, it still has a negative 2nd derivative here, and so is unstable: photons can loiter near $R = 3$ but will eventually move away from it. The photon sphere made the headlines in 2019 with the 'Event Horizon Telescope' image of the emission around the black hole in M87. This was modelled by assuming that most of the observed radiation came from photons that were 'leaking' away from the photon sphere around the $10^{12} M_\odot$ black hole at the centre of that galaxy.

7.2 Binding energy and accretion efficiency

The use of a Newtonian analogy for effective potential and total energy can be made more precise. In the Lagrangian formalism, the lack of an explicit dependence on a coordinate leads to a conserved quantity: if $\partial L/\partial q = 0$, the 'momentum' $\partial L/\partial \dot{q}$ is constant. In the non-relativistic formalism, time is not a coordinate as such – but a time-independent Lagrangian still leads to the Hamiltonian as a constant of the motion. In GR, time is a coordinate, so there is an energy-like conserved quantity when there is no explicit time dependence: $\partial L/\partial \dot{t} =$ constant. This derivative is $(1/2L)\partial L^2/\partial \dot{t}$ or $(1/2c)\partial L^2/\partial \dot{t}$. Since $L^2 = g_{\mu\nu}U^\mu U^\nu$, we have $\partial L^2/\partial \dot{t} = 2cg_{0\mu}U^\mu = 2cU_0$. In the SR limit, then, $\partial L/\partial \dot{t}$ is just γc, which is $1/c$ times the energy per unit mass. For the Schwarzschild case, $\partial L/\partial \dot{t} = \alpha c \dot{t}$ and hence we can identify $k = \alpha \dot{t}$ as giving E/mc^2: the ratio of the total energy to the the total rest-mass energy. In our previous analysis, we had $L^2 = c^2 = c^2\alpha(k/\alpha)^2 - \dot{r}^2/\alpha - r^2(h^2/r^4)$, so for a circular orbit with $\dot{r} = 0$

$$\alpha\,\frac{h^2}{r^2} = c^2(k^2 - \alpha) \;\Rightarrow\; k = \sqrt{\alpha(1 + h^2/r^2c^2)}. \tag{98}$$

Putting in the criteria for the innermost stable orbit ($R = 6$; $H^2 = 12$), we get

$$k = \sqrt{8/9}, \tag{99}$$

and so a particle at the innermost stable orbit has a total energy that is about 6% less than its rest-mass energy. To have reached this point, the particle must have converted the missing energy into another form. In practice, material around a black hole will probably settle into an **accretion disc** that is flattened by rotation. The different orbital angular velocities at different radii will generate frictional heating of the disc as material slowly spirals inwards – and eventually the liberated energy escapes as radiation from the disc. It is normally assumed that the disc has an inner edge at the innermost stable orbit, beyond which material is rapidly captured by the black hole. Thus in summary accretion of material onto a Schwarzschild black hole has the potential to liberate 6% of the accreted mass-energy as radiation; even higher efficiencies are possible if we do the same calculation for the **Kerr metric**, which corresponds to a black hole endowed with angular momentum. This calculation has an obvious relevance to the energy release from active galaxies, where there is a central supermassive black hole

7.3 Advance of the perihelion of Mercury

Having seen the types of orbits we can find in a Schwarzschild spacetime, we now apply this to the Solar System. This was first done by Einstein in 1916 for Mercury, in order to see if GR could account for an apparent peculiarity of its orbit. We begin with a simple version of the argument, using the radial force equation $\ddot{r} = -dV_{\rm eff}/dr$. Suppose we have an orbit that is very nearly circular, $r = r_c + \epsilon$: then we can approximate the potential using a Taylor series:

$$\ddot{\epsilon} = -\frac{d^2V_{\rm eff}}{dr^2}\,\epsilon, \tag{100}$$

which shows that the radius undergoes harmonic oscillations, with angular frequency given by $\omega^2 = d^2V_{\rm eff}/dr^2$. The second derivative is easy to evaluate, generating three terms. One of these can be eliminated using $dV_{\rm eff}/dr = 0$, to yield

$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{h^2}{r^4}\left(1 - \frac{6GM}{c^2r}\right). \tag{101}$$

This says that the orbit is almost an ellipse. For a Newtonian circular orbit with angular frequency ω, we have $r^2\omega = h$, from the definition of h plus the fact that ω *means* $\dot{\phi}$, so $\omega^2 = h^2/r^4$ and the orbital and radial oscillation frequencies would be the same if the $6GM/c^2r$ term were absent or negligible – as it would be at large r. In that case, the equality of radial and angular frequencies means that the orbit closes and keeps a fixed elliptical shape. But the perturbation reduces the frequency of the radial oscillations, so the orbit has to make more than one rotation between two occurrences of **perihelion** (closest point to the Sun). Thus we have a **rosette orbit** that undergoes **precession**. Since the fractional change in radial frequency is $-3GM/c^2r$ the change in phase per orbit, $\delta\omega \times (2\pi/\omega)$ can be written as

$$\boxed{\Delta\phi = \frac{6\pi GM}{c^2r}\ \text{radians per orbit.}} \tag{102}$$

This is for a very nearly circular orbit. For a substantially elliptical orbit, we have to work a bit harder. It is convenient to start by changing variable to $u \equiv 1/r$. We can eliminate time in favour of ϕ as the parameter describing the orbit by using angular momentum:

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = -\frac{1}{u^2}\frac{du}{d\phi}hu^2 = -h\frac{du}{d\phi}. \tag{103}$$

The geodesic equation (93) then becomes

$$\frac{h^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{h^2u^2}{2} - GMu - \frac{GM}{c^2}h^2u^3 = c^2(k^2-1)/2. \tag{104}$$

Differentiating w.r.t. ϕ and dividing by $du/d\phi$ gives

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2.} \tag{105}$$

In Newtonian physics the last term would be absent, giving the equation of an elliptical orbit: $u(\phi) = (GM/h^2)(1+e\cos\phi)$, where e is the **eccentricity** or **ellipticity** of the orbit and $\phi = 0$ is chosen as the point of perihelion.

The last GR correction term is very small for Mercury's orbit, $\sim 10^{-7}$ of GM/h^2, so it can be treated as a perturbation. First make things dimensionless by defining the radius in terms of the circular Newtonian radius for the same h:

$$U \equiv u\frac{h^2}{GM}. \tag{106}$$

So the equation of motion is now

$$\frac{d^2U}{d\phi^2} + U = 1 + \epsilon U^2; \qquad \epsilon \equiv 3G^2M^2/(c^2h^2). \tag{107}$$

We can now expand the solution as $U = U_0 + U_1$, where U_0 is the Newtonian solution for $\epsilon \to 0$: $U_0 = 1 + e\cos\phi$. The perturbation U_1 must be $O(\epsilon)$, so to linear order in ϵ, ϵU^2 can be replaced by ϵU_0^2 in the equation of motion. Subtracting the unperturbed $d^2U_0/d\phi^2 + U_0 = 1$, we get an equation for U_1:

$$\frac{d^2U_1}{d\phi^2} + U_1 = \epsilon U_0^2 = \epsilon\left(1 + 2e\cos\phi + e^2\cos^2\phi\right) = \epsilon\left(1 + 2e\cos\phi + \frac{e^2}{2} + \frac{e^2}{2}\cos 2\phi\right). \tag{108}$$

The complementary function gives nothing new ($\propto U_0$) and the solution is

$$U_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{109}$$

(extra ϕ because $\sin\phi$ is in the complementary function). The solution (exercise) is

$$U_1 = \epsilon\left[\left(1 + \frac{e^2}{2}\right) + e\phi\sin\phi - \frac{e^2}{6}\cos 2\phi\right]. \tag{110}$$

Ignoring everything except the growing term $\propto \phi$, we find

$$\begin{aligned} U &\simeq 1 + e\cos\phi + \epsilon e\phi\sin\phi \\ &\simeq 1 + e\cos\left[\phi\left(1-\epsilon\right)\right], \end{aligned} \tag{111}$$

to $O(\epsilon)$. Thus the orbit is periodic, with period (in ϕ) of

$$\frac{2\pi}{1-\epsilon}. \tag{112}$$

Hence the perihelion moves forward through an angle $2\pi\epsilon$ per orbit, as before. Notice that, having gone to all this extra trouble to be able to handle the case of highly elliptical orbits, the answer we obtain is independent of e when expressed in terms of M and h, and is identical with our simple argument for the $e \ll 1$ case.

The advance of the perihelion of Mercury was a known problem from about 1859 and solutions were sought using classical celestial mechanics. The total observed effect is very nearly 5600 arcseconds per century (1 arcsecond = $1/3600^{\rm th}$ of a degree). Most of this (5026) arises from the precession of the Earth's spin axis. The next largest contribution is gravitational perturbations from other planets, which contribute about 531 arcseconds per century. However, the observations showed that there was an additional 43 arcseconds per century remaining. Attempts were made to account for this with a new planet inside Mercury's orbit (Vulcan), but this was never seen. The discrepancy was finally explained by Einstein in 1916. For Mercury's orbit of $T = 88$ days, $r = 5.8 \times 10^{10}$ m and $e = 0.2$, the GR prediction is that the advance of Mercury's perihelion is 43 arcseconds per century. This spectacular agreement did much to establish the credibility of the new theory.

One can however note in passing that what this mainly establishes is the existence of an extra term in the potential $\propto 1/r^3$. This is of the form of a quadrupole, which would arise if the Sun was flattened – and it doesn't need to be flattened by very much. So skeptics might not have been satisfied. But GR shows much larger deviations from Newtonian behaviour when it comes to light, as we now demonstrate.

7.4 The bending of light around the Sun

In addition to the precession of the perihelion of Mercury, and gravitational redshifting, there are two other 'classic' tests of GR. The first of these is the famous bending of light around the Sun by Gravitational Lensing. That gravity should bend light can be understood qualitatively from the Equivalence Principle. Imagine we observe a light beam travelling in a straight line in a freely-falling laboratory that defines a LIF. Now accelerate the laboratory and observe the same light beam: the increasing velocity of the observer will alter the apparent direction of the light beam – the familiar phenomenon of **aberration of light** (first used to prove that the Earth moves, by James Bradley in 1727). Thus the light beam appears to take a curved path in the accelerating laboratory; by the Equivalence Principle, the same must hold in a gravitational field.

But getting the right magnitude for this bending is not so easy. Consider first the following Newtonian argument (which was how Einstein first reasoned, in about 1912). Treat the photon as massive particle travelling at c, and apply the EP in the form "all objects fall equally fast". For the present purpose, we are interested in light deflection, so we think about the component of gravitational acceleration perpendicular to the photon path, $a_\perp$. The EP suggests that this will change the perpendicular momentum of the photon at the same rate as for any particle: $dp_\perp/dt = a_\perp(E/c^2) = a_\perp(p/c)$, where p is the total photon momentum. So to calculate the angle by which light is deflected in travelling along some path, we just need to integrate to get the total perpendicular momentum acquired, and then the deflection angle is

$$\theta = \frac{\delta p_\perp}{p} = \frac{1}{p}\int \frac{dp_\perp}{dt}\, dt = \int a_\perp\, dt/c \tag{113}$$

(assuming the deflection angle to be small, which is normally the case). For small deflections, the integral can be evaluated **impulsively** using the **Born approximation**, i.e. assuming that the photon follows some straight unperturbed trajectory and assuming that the acceleration is very little changed on the nearby exact path. Let's apply this reasoning to deflection by a point mass M, for a path with **impact parameter** (distance of closest approach) R. If x is a coordinate along the path with $x = 0$ at closest approach, then $r^2 = x^2 + R^2$, and $a_\perp = a \sin\phi$, where $\sin\phi = R/r$ (see Figure 3). Then $a_\perp = (GM/r^2)(R/r)$, so

$$\theta = \int_{-\infty}^{\infty} (GM/r^2)(R/r)\, dx/c^2 = \frac{GM}{c^2 R}\int R^2/r^3\, dx = \frac{GM}{c^2 R}\int \frac{dy}{(1+y^2)^{3/2}}, \tag{114}$$

where $y = x/R$. The last integral is equal to 2, so we predict

$$\theta = \frac{2GM}{c^2 R}. \tag{115}$$

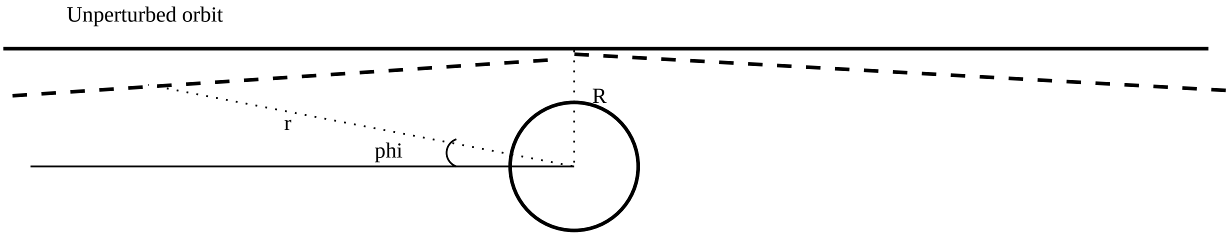

Figure 3: *Light bending round the Sun.*

Now, it turns out that this estimate is too low by exactly a factor 2. It was the verification of this factor 2 by an expedition led by Sir Arthur Eddington in 1919 that convinced the scientific community overnight that GR was correct and Newtonian gravity had to be abandoned. Eddington's team photographed a total eclipse of the Sun and measured the angular distance through which the Sun moved the apparent position of nearby stars. Putting the Sun's mass and radius into our formula, and multiplying by two, this deflection is about 1.75 arcseconds – quite a challenge to detect with the available small telescopes that had to be transported to exotic locations.

There is a relatively quick and illuminating way of seeing how this factor 2 arises. We are interested in the limit of weak gravitational fields, where the Schwarzschild metric (equation 84) looks as follows:

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{116}$$

where $\Phi = -GM/r$. A peculiar thing about this is that the spatial curvature seems confined to the radial direction – but as we discussed previously, this is to do with our choice of radial coordinate, which is a definition that can be changed. Suppose we put $r' = r(1 + \Phi/c^2)$: because $r\Phi$ is a constant, $dr' = dr$. But the change of radial coordinate puts the spatial metric into the form of $(1 - 2\Phi/c^2)$ times Euclidean space. So we can abandon spherical polars and write the weak-field metric in the **isotropic form**:

$$\boxed{c^2 d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \left(dx^2 + dy^2 + dz^2\right).} \tag{117}$$

In fact, this metric applies for *any* static weak-field gravitational field, not just the Schwarzschild case. When we consider this metric for light with $d\tau = 0$, we see that the **coordinate speed** of light varies with position:

$$\left|\frac{d\mathbf{x}}{dt}\right| = c\left(1 + \frac{2\Phi}{c^2}\right). \tag{118}$$

In effect, spacetime behaves as a kind of glass, with a varying refractive index, and leading directly to light deflection as in optics. But what we can notice is that the perturbation to the effective c gains equal contributions from the time and space parts of the metric. Our equivalence-principle argument was based on our previous Newtonian experience where g_{00} supplied the gravitational force, but this was flawed because non-relativistic particles are not sensitive to the spatial part of the metric. Including the effects of these here clearly doubles the effect we would have had if we had considered g_{00} only, making it inevitable tha our quasi-Newtonian guess for the deflection angle would be too low by a factor 2.

We now derive the GR light bending in more detail, using the geodesic equation for a massless photon. As before, the proper time for the photon is zero ($d\tau = 0$), so we shall use affine parameter p rather than proper time τ. For massless particles $L^2 = 0$, and so the ELII form still holds. As before

$$\begin{aligned} \alpha \dot{t} &= k = \text{constant}, \\ r^2 \dot{\phi} &= h = \text{constant}, \end{aligned} \tag{119}$$

where $\dot{} = d/dp$. For light $L^2 = 0$ so

$$0 = \alpha c^2 \dot{t}^2 - \frac{\dot{r}^2}{\alpha} - r^2 \dot{\phi}^2. \tag{120}$$

Rearranging, and writing in terms of $u = 1/r$ as before,

$$h^2 \left(\frac{du}{d\phi}\right)^2 = c^2 k^2 - \alpha h^2 u^2 = c^2 k^2 - h^2 u^2 + \frac{2GM}{c^2} h^2 u^3. \tag{121}$$

Differentiating as before,

$$\boxed{\frac{d^2 u}{d\phi^2} + u = \frac{3GM}{c^2} u^2.} \tag{122}$$

We treat the RHS as a perturbation to the straight-line orbit $u_0 = (\sin\phi)/R$, where R is the distance of closest approach. Letting $u = u_0 + u_1$,

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \frac{3GM}{c^2 R^2} \sin^2\phi = \frac{3GM}{2c^2 R^2}(1 - \cos 2\phi). \tag{123}$$

By inspection, the first-order solution is

$$u = \frac{\sin\phi}{R} + \frac{3GM}{2c^2 R^2}\left(1 + \frac{1}{3}\cos 2\phi\right). \tag{124}$$

At large distances, where $u = 0$, ϕ is small, $\sin\phi \simeq \phi$, and $\cos 2\phi \simeq 1$, so

$$u_{-\infty} = 0 \Rightarrow \frac{\phi_{-\infty}}{R} + \frac{2GM}{c^2 R^2} = 0 \Rightarrow \phi_{-\infty} = -\frac{2GM}{c^2 R}, \tag{125}$$

where $\phi_{-\infty}$ is the angle of the incoming photon from $r \to -\infty$. So the incoming photon comes in at a slight angle due to the slight repulsion of the general relativistic correction to the angular momentum term, arising from the distortion of radial distance. Similarly, after the light has passed the source, it reaches infinite distance at

$$\phi_{+\infty} = \pi + \frac{2GM}{c^2 R}. \tag{126}$$

The total deflection is then

$$\Delta\phi_{\rm GR} = \frac{4GM}{c^2 R}, \tag{127}$$

verifying the expected factor of 2 increase with respect to the Newtonian result.

7.5 Time delay of light

The final classical test of GR was proposed by Irwin Shapiro in 1964 and subsequently measured by him in 1966. The idea is to bounce a radar signal off Venus and measure the time for it to return. As we saw in discussing light deflection, the coordinate speed of light is slowed by the

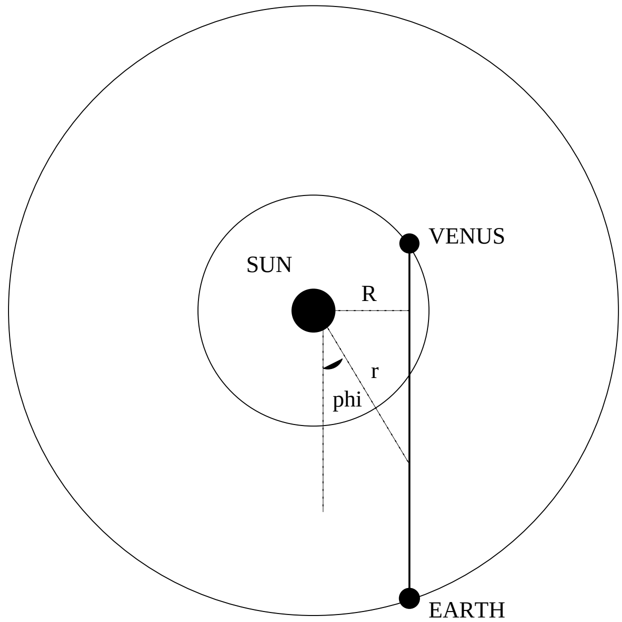

Figure 4: Geometry of the radar time delay between Earth and Venus.

Sun's gravitational field, so the radar pulse takes longer to travel than we would expect from the geometry of the situation. This is illustrated in Figure 4. Much of the effect comes from light rays that pass close to the Sun, so we will assume henceforth that the experiment involves Venus being on the far side of the Sun, so that the distance of closest approach, R, can be almost as small as the radius of the Sun. Because the fractional change in light speed is $2GM/c^2r$, the accumulated time delay is (to linear order in M):

$$\Delta t = \frac{2GM}{c^2 r}\,\frac{dx}{c}, \tag{128}$$

where x is a coordinate that runs between the two planets. It is related to the radius by $r^2 = R^2 + x^2$, so that $dx = r\,dr/\sqrt{r^2 - R^2}$. Hence the time delay is (multiplying by 2 to allow for the outward and return journey):

$$\Delta t = \frac{4GM}{c^3}\left(\int_R^{r_e} \frac{dr}{\sqrt{r^2 - R^2}} + \int_R^{r_v} \frac{dr}{\sqrt{r^2 - R^2}}\right), \tag{129}$$

where r_e and r_v are the orbital radii of Earth and Venus. The integrals are $\ln(\sqrt{(r_e/R)^2 - 1} + r_e/R)$ and similar for Venus, which become just $\ln(2r_e/R)$ when $r_e \gg R$. Hence a reasonable approximation for the total delay is

$$\Delta t \simeq \frac{4GM}{c^3}\,\ln\left(\frac{4r_e r_v}{R^2}\right). \tag{130}$$

This is the dominant effect, but there are other terms of order GM/c^3. One comes because this is the elapsed **coordinate** time, whereas we want the elapsed proper time on Earth:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{r_e c^2}\right) c^2 dt^2 - r_e^2 d\phi^2 \tag{131}$$

(treating orbits as circular, so $dr = 0$). Hence the proper time elapsed on Earth is

$$\tau = \sqrt{\alpha(r_e) - \frac{r_e^2}{c^2}\left(\frac{d\phi}{dt}\right)^2}\; t. \tag{132}$$

The angular velocity of the Earth obeys $(d\phi/dt)^2 = GM/r_e^3$ (a Newtonian approximation is good enough here), so the time conversion is

$$\frac{\tau}{t} = \sqrt{1 - \frac{2GM}{r_e c^2} - \frac{GM}{r_e c^2}} = \sqrt{1 - \frac{3GM}{r_e c^2}}. \tag{133}$$

Since the total journey time is $\sim r_e/c$, the extra offset is $\sim GM/c^3$, which is negligible in comparison with the main log term.

It may also be objected that we have used the Born approximation and integrated along an unperturbed trajectory, whereas the true path will be bent. We now repeat the analysis using the exact geodesic equations. As before, we have the t and ϕ equations $\alpha\dot{t} = k$ and $r^2\dot{\phi} = h$. The $L^2 = 0$ 'energy' equation is

$$\alpha c^2 \dot{t}^2 - \dot{r}^2/\alpha - r^2\dot{\phi}^2 = 0 \Rightarrow \dot{r}^2 + \alpha h^2/r^2 = k^2 c^2. \tag{134}$$

Now, at perihelion, $\dot{r} = 0$, so $k^2c^2 = h^2\alpha(R)/R^2$ and hence k can be eliminated from the equation for $\dot{r}$. Finally, we can use the t equation to convert to $dr/dt = \dot{r}/\dot{t}$:

$$\frac{dr}{dt} = \pm\frac{\alpha(r)Rc}{\alpha(R)^{1/2}}\left(\frac{\alpha(R)}{R^2} - \frac{\alpha(r)}{r^2}\right)^{1/2}, \tag{135}$$

an equation that is pleasingly independent of k and h. For the present purpose, it will suffice to expand this to get the effect to first order in M, which with a bit of work (or the aid of *Mathematica* or equivalent) comes out as

$$\frac{dt}{dr} = \frac{r}{c\sqrt{r^2 - R^2}} + \frac{(2r + 3R)}{(r + R)\sqrt{r^2 - R^2}}\,\frac{GM}{c^3}. \tag{136}$$

Integrating from R to r, we get

$$\Delta t = \frac{\sqrt{r^2 - R^2}}{c} + \left(\frac{\sqrt{r^2 - R^2}}{r + R} - 2\ln R + 2\ln(r + \sqrt{r^2 - R^2})\right)\frac{GM}{c^3}. \tag{137}$$

The first term is the Newtonian distance. The GR correction for $r \gg R$ is dominated by a $2\ln(2r/R)\, GM/c^3$ term. This is for a one-way trip from R to one of the planets. Allowing for two planets and a 2-way trip, we get $4\ln(4r_1r_2/R^2)\, GM/c^3$ as before, so the total delay is dominated by the reduced speed of light.

When R is as small as possible (the radius of the Sun), the total delay is around 200μs. With 1960s equipment, this could be verified to about 5% precision. Today, using Solar System spacecraft, the agreement with GR has been shown to work at the 0.002% level.

8 Mathematical foundations of GR

So far this course has taken a deliberately informal approach, with the aim of emphasising that it is possible to get quite a long way into GR using the same concepts and mathematical tools that we employ in SR. But there is a danger in this approach of failing to build a sufficiently secure foundation. Having gained some first feeling for how GR works, it is time to regroup and think more generally about the concepts and the mathematics needed to tackle curved spacetime, setting in place the tools that will let us continue the journey of GR through to Einstein's gravitational field equations and their applications.

Our main aim will be to extend the SR language of 4-vectors into the tools of tensor calculus. This presents a challenge of notation, since more mathematically inclined texts prefer a 'coordinate-free' approach that can look very different to the more traditional notation – so much so that different books on GR can seem at first sight almost to refer to different subjects. The problem is that our approach so far has been insufficiently *geometrical.* In ordinary vector algebra, we are familiar with the idea that a vector is an object with a magnitude and direction, which exists independent of the coordinates used to describe it. In order to turn a vector into a specific set of numbers, $\mathbf{v} = (v^1, v^2, v^3)$, we first have to choose a set of basis vectors, $\mathbf{e}_i$, and express the vector as a superposition of these:

$$\mathbf{v} = \sum_i v^i \mathbf{e}_i. \tag{138}$$

We can decide to use a different basis, and then the components v^i will change – so the vector is definitely more than just a set of components. Vector operations also have a geometrical significance, with the dot product projecting one vector along another, and the cross product describing the area of the parallelogram described by two vectors. All this is missed if we simply write down the formulae for these quantities in terms of components. And yet practical computations do require us to work in this way; for this reason, much of the focus in the treatment of GR here will be on component-based formalism. But we will try to illuminate this material with a geometric viewpoint, both in the hope of providing a deeper understanding of the key mathematical concepts being used, and also to make it possible to consult a wider range of textbooks. For those who wish to take this approach further, an excellent reference is Schutz *Geometrical methods of mathematical physics* (CUP).

8.1 Manifolds and vectors

The branch of mathematics that discusses vectors in a general coordinate-free fashion is **differential geometry**, and its most fundamental concept is the **differentiable manifold**. Informally, a manifold is a space where the points can be parameterized continuously and differentiably as an n-dimensional set of real numbers, x^i, which are the coordinates of the point. This means that points can be connected by a curve governed by some parameter along the curve, λ, and that the derivatives of the coordinates $dx^i/d\lambda$ exist. Euclidean space is a manifold, but it is a special one because a manifold need not possess a metric structure, i.e. any notion of distance between points. Nevertheless, because of its differentiable nature, a manifold can be thought of *locally* as being close to Euclidean space.

8.1.1 Tangent vectors and tangent spaces

Given a manifold, how is the general analogue of Euclidean vector algebra generated? How do we even define vectors, and how are we to write them as a sum over basis vectors? In a manifold, points have parameterized curves passing through them, and this can be used to generate **tangent vectors** $d/d\lambda$, where

$$\boxed{\frac{d}{d\lambda} = \sum_i \frac{dx^i}{d\lambda}\,\frac{\partial}{\partial x^i}.} \tag{139}$$

This is an operator relation that applies to any function $f(x^i)$, and is just obtained from the chain rule. This may not look like it has much to do with directions and magnitudes, but $d/d\lambda$ is a vector in the sense described above, where $\partial/\partial x^i$ is the basis. A sum of two vectors $a(d/d\lambda) + b(d/d\mu)$ (where a and b are scalars) is just a sum over the basis with a different set of coefficients, and so looks like $d/d\nu$, where ν is the parameter of another curve passing through the point of interest. So now we have vectors on a manifold, and can manipulate them in the usual vector algebra of addition. The vectors are defined locally, so they inhabit not the full manifold, but a **tangent space**. This sounds a rather abstract distinction; we will make it more concrete below.

8.1.2 Differential forms and tensors

The next concept needed is some analogue of the magnitude of a vector – that is, we need something that will act as a function of a vector to produce a scalar. This entity is called a **1-form**, and is defined by the way it acts on vectors:

$$\tilde{\boldsymbol{\omega}}(\mathbf{v}) = \text{scalar}. \tag{140}$$

The function is linear with respect to addition of vectors and multiplication by scalars:

$$\begin{aligned} \tilde{\boldsymbol{\omega}}(a\mathbf{v} + b\mathbf{w}) &= a\tilde{\boldsymbol{\omega}}(\mathbf{v}) + b\tilde{\boldsymbol{\omega}}(\mathbf{w}) \\ (a\tilde{\boldsymbol{\omega}} + b\tilde{\boldsymbol{\sigma}})(\mathbf{v}) &= a\tilde{\boldsymbol{\omega}}(\mathbf{v}) + b\tilde{\boldsymbol{\sigma}}(\mathbf{v}). \end{aligned} \tag{141}$$

This latter equation suggests a symmetry: the vector $\mathbf{v}$ can equally well be regarded as operating on the 1-form $\tilde{\boldsymbol{\omega}}$. We say that the objects are **dual**:

$$\tilde{\boldsymbol{\omega}}(\mathbf{v}) = \mathbf{v}(\tilde{\boldsymbol{\omega}}) \equiv \langle \tilde{\boldsymbol{\omega}} | \mathbf{v} \rangle. \tag{142}$$

A pictorial way to visualise 1-forms is to think of a vector as an arrow, indicating direction and magnitude, while a 1-form is a series of surfaces, similar to contours on a map. The number formed by contracting the two is the number of contours pierced by the vector arrow; thus a 1-form of larger magnitude corresponds to a closer contour spacing.

The concept of vectors as machines that 'eat' a 1-form to produce a number, or vice-versa, can be immediately generalized to **tensors** that may require several meals of each type in order to yield a scalar. An (n, m) tensor T is an object that operates on n 1-forms and m vectors in order to produce a number:

$$\boxed{T(\tilde{\boldsymbol{\omega}}_1, \tilde{\boldsymbol{\omega}}_2, \ldots, \tilde{\boldsymbol{\omega}}_n, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) = \text{scalar}.} \tag{143}$$

8.1.3 The metric tensor

In GR, the most important tensor is the **metric tensor**, which is a symmetric 2-form that yields a scalar when given two vectors:

$$\boxed{g(\mathbf{v}, \mathbf{u}) = g(\mathbf{u}, \mathbf{v}) \equiv \mathbf{u} \cdot \mathbf{v}\ .} \tag{144}$$

The metric thus provides a way of associating a 1-form $g(\mathbf{v},\ \ )$ with a vector, and defines the notion of an **inner product** between vectors. It therefore gives a 'measuring ruler' for determining the 'length' of a vector, though the quantity $\sqrt{g(\mathbf{v}, \mathbf{v})}$. Not all manifolds possess a metric structure, but if we have a **Riemannian manifold** where a metric does exist, it simplifies the discussion.

8.1.4 Tangent spaces and embedding

Where we have a metric, it gives a more concrete meaning to a tangent space, which was defined rather abstractly as a 'container' for the tangent vectors. Now we can ask if there exist a set of coordinates where the manifold is locally Euclidean, in the sense that the metric at a point defines a Euclidean distance between points, $ds^2 = dx^2 + dy^2 + \cdots$ and also that the first derivatives of the metric vanish. We include here the pseudo-Euclidean case with altered signature, as needed to accommodate relativity. We will not prove it here, but it is always possible (see e.g. section 2.11 of Hobson, Efstathiou & Lasenby) to choose coordinates in this way. When it is done, these coordinates define the tangent space. We have already met this phenomenon in GR through the equivalence principle, where the tangent space is a local inertial frame: the metric is that of special relativity and its derivatives vanish, as required in order for there to be no gravitational forces.

Some symmetric Riemannian manifolds can conveniently be described as **embedded** in a Euclidean space of one higher dimension. Vectors can then be defined as usual via Cartesian coordinates in the $(D+1)$-dimensional space. In this picture, the tangent space justifies its name, and is a D-dimensional subspace. This simple embedding is not always possible, although Nash (of *A Beautiful Mind* fame) showed that embedding is always possible – but may require a Euclidean space of up to 230 dimensions for a general 4D manifold. A familiar simple example of embedding would be the surface of a sphere of fixed radius R: the $\mathbf{r} = (x, y, z)$ coordinates of a point on the sphere as a function of the 2D coordinates on the manifold can be written down using spherical polars at constant radius. In this embedded case, coordinate basis vectors are just

$$\mathbf{e_i} = \frac{\partial \mathbf{r}}{\partial c^i}, \tag{145}$$

where c^i is the ith coordinate. This follows the general expression above where we saw that a coordinate basis vector was supplied by $\partial/\partial c^i$.

8.2 Components of vectors and 1-forms

If we expand a vector in (contravariant) components, $\mathbf{v} = a^i \mathbf{e}_i$, then the linearity of the 1-form function can be used to write $\tilde{\boldsymbol{\omega}}(\mathbf{v}) = a^i b_i$, where we expand the 1-form in the same way as for the vector: $\tilde{\boldsymbol{\omega}} = b_i \tilde{\mathbf{e}}^i$, defining a **dual basis**

$$\tilde{\mathbf{e}}^i(\mathbf{e}_j) = \delta^i_j. \tag{146}$$

Note that we naturally introduce upstairs and downstairs indices to distinguish things associated with vectors and 1-forms. Given a set of coordinates x^i, we can write a vector $d/d\lambda$ as components $dx^i/d\lambda$ combining with basis vectors

$$\mathbf{e}_i = \frac{\partial}{\partial x^i}. \tag{147}$$

It is also possible to conceive of sets of basis vectors not derived from a set of coordinates (although these are clearly harder to construct); the above basis is therefore called a **coordinate basis**. Once we have a basis, components are defined by operating on the members of the appropriate basis with a vector or a 1-form:

$$\begin{aligned} a^i &= \mathbf{v}(\tilde{\mathbf{e}}^i) \\ b_i &= \tilde{\boldsymbol{\omega}}(\mathbf{e}_i). \end{aligned} \tag{148}$$

We will not discuss how the dual basis 1-forms might be constructed in general, but will content ourselves with the case where a metric exists. In effect, $g(\mathbf{e}_\mu, \quad)$ defines a basis 1-form $\tilde{\mathbf{e}}^\mu$. This is not a dual basis (unless the metric is diagonal), but it can be made so via appropriate linear combinations.

Similarly, the components of a tensor T are derived by inserting n basis 1-forms and m basis vectors in the appropriate slots of T, and as a result are written with n upstairs indices and m downstairs indices. An important case is the components of the metric. The metric is also defined to be **bilinear**: linear in each argument, and so

$$g(\mathbf{v},\mathbf{u}) = g(\mathbf{e}_\mu,\mathbf{e}_\nu)\, v^\mu u^\nu \equiv g_{\mu\nu}\, v^\mu u^\nu, \tag{149}$$

which defines the components of the metric and shows how they combine with vector components to make a scalar invariant.

8.2.1 Basis transformations

Only at this stage is it necessary to introduce the concept of coordinate transformations that is so central to traditional tensor calculus. We approach this by considering a basis transformation, which should be linear so that the new basis can still be a complete set:

$$\mathbf{e}'_j = \alpha^i_j \mathbf{e}_i. \tag{150}$$

The dual basis will have an analogous transformation:

$$\tilde{\mathbf{e}}'^j = \beta^j_i \tilde{\mathbf{e}}^i, \tag{151}$$

where we have to maintain duality:

$$\delta^i_j = \tilde{\mathbf{e}}'^i(\mathbf{e}'_j) = \alpha^k_j \beta^i_\ell \delta^\ell_k = \alpha^k_j \beta^i_k. \tag{152}$$

Thus the α and transposed β matrices are inverses of each other. For the simplest case of a coordinate basis, we saw that the basis vectors were $\mathbf{e}_i = \partial/\partial x^i$, so the transformation coefficients are just derivatives between the new and old coordinates:

$$\alpha^i_j = \frac{\partial x^i}{\partial x'^j}; \quad \beta^i_j = \frac{\partial x'^i}{\partial x^j}. \tag{153}$$

Since the vector is not changed by the basis transformation, the new components are

$$v'^j = \tilde{\mathbf{e}}'^j(\mathbf{v}) = \beta^j_k \tilde{\mathbf{e}}^k(\mathbf{v}) = \beta^j_k v^k, \tag{154}$$

which is opposite to the transformation law for basis vectors and justifies the name *contra*variant component. Similarly, the components of 1-forms transform with the α matrix, in the same way as the basis vectors.

This gives us a clearer idea of why we needed to introduce covariant and contravariant components in our earlier discussion: our previous approach missed the key point of the distinct geometrical roles of a vector and a 1-form, each of which has a separate set of components. This distinction is obscured if we insist on thinking of covariant and contravariant components as two equivalent ways of describing the same vector. Furthermore, the conventional discussion of covariant and contravariant vectors in relativity relies on converting one into the other via the metric tensor, but we have seen the surprising fact that a dual structure of vectors and 1-forms can still exist even without the existence of a metric.

8.2.2 Affine geometry

A related concept that often arises in discussions of relativity is that of **affine geometry**. Consider a set of coordinates over some manifold, and write these in a form that resembles a

Euclidean position vector, $\mathbf{v} = (x, y, z, \cdots)$. The **affine transformation** of the coordinates can be written is the general linear change $\mathbf{v}' = \mathbf{A} \cdot \mathbf{v} + \mathbf{b}$, i.e. a translation in combination with rotation, shear, and scale change, all specified by the matrix $\mathbf{A}$. The resulting **affine space** is more general than a vector space because the translation means that there is no preferred origin, whereas a vector space has a zero element that combines linearly with a vector to leave it unchanged. But conversely, affine transformations are not completely general because of the requirement of linearity: Lorentz transformations in Special Relativity are special cases of affine transformations, but general coordinate transformations are not. However, locally these transformations are linear, because of the requirement that the manifold must be differentiable; thus differential geometry is inevitably affine in nature.

A further point of interest is that affine spaces do not require the existence of a metric: indeed, any initial 'distance' defined between two points can be changed after the transformation. Nevertheless, the transformation keeps parallel lines parallel, so that one can think of distance along a line and compare distances along different lines that are parallel. A good example of an affine space without a metric arises in colour vision: the basis vectors are the different colours and the v_i are the intensities in each of red, green and blue; one can compare overall intensities when the colour balance is unaltered, but this is not meaningful for different colour mixtures.

8.3 Areas and antisymmetric tensors

The 1-form is in fact a special case of an important class of tensors, called **differential forms**, which are antisymmetric in all their arguments. The reason for wanting to introduce such objects is primarily *geometrical*: it is often necessary to be able to deal algebraically with the area (i.e. the parallelogram) defined by two vectors. This must clearly be represented by a $(0, 2)$ tensor, since the magnitude of the area is defined as a bilinear function of two vectors: area$(\mathbf{a}, \mathbf{b})$. The antisymmetry comes about by the reasonable requirement that area$(\mathbf{a}, \mathbf{a}) = 0$; write $\mathbf{a} = \mathbf{b} + \mathbf{c}$ and bilinearity then requires

$$\text{area}(\mathbf{b}, \mathbf{c}) = -\text{area}(\mathbf{c}, \mathbf{b}). \tag{155}$$

This naturally generalizes to volumes of parallelepipeds, leading to the introduction of n-forms of arbitrary order. Symmetric terminology then leads us to define an n-vector as a totally antisymmetric $(n, 0)$ tensor. A mechanism for generating these higher-order forms is provided by the **wedge product**:

$$\boxed{\tilde{\mathbf{a}} \wedge \tilde{\mathbf{b}} \equiv \tilde{\mathbf{a}} \otimes \tilde{\mathbf{b}} - \tilde{\mathbf{b}} \otimes \tilde{\mathbf{a}},} \tag{156}$$

where the **direct product** or **outer product** is defined by

$$\tilde{\mathbf{a}} \otimes \tilde{\mathbf{b}}(\mathbf{c}, \mathbf{d}) \equiv \tilde{\mathbf{a}}(\mathbf{c})\, \tilde{\mathbf{b}}(\mathbf{d}). \tag{157}$$

This generalizes to larger wedge products $\tilde{\mathbf{a}} \wedge \tilde{\mathbf{b}} \wedge \tilde{\mathbf{c}} \ldots$ in an obvious way.

In index terms, the non-zero components of an n-form in m dimensions, $\omega_{ijk\ldots}$, must have a set of indices $ijk\ldots$ that form a distinct combination drawn from $1, \ldots m$. Thus, there can be only ${}^mC_n$ independent components and there exists only one m-form in m dimensions. This number of components is in general different to the number of dimensions; since ${}^3C_2 = 3$, however, the area defined by two vectors can be associated with a vector in three dimensions. The vector cross product is a piece of machinery constructed to exploit this special case, and cannot be defined in other numbers of dimensions. But we now see that it is a special case of the wedge product, which defines an orientated area in any number of dimensions (just a single number in 2D).

9 Tensor analysis in component form

Having gained some important geometrical insights, we must now acknowledge that most practical tasks in tensor analysis will involve coordinate systems and components. So we now take stock of the necessary tools from this point of view.

The justification for using tensors is the same as wanting to write relativistic equations in terms of 4-vectors: to maintain **general covariance**. As we saw, a 4-vector equation $A^\mu = B^\mu$ is guaranteed to hold in all frames, because both sides of the equation change in the same way under coordinate transformations. But not all physical laws can be written in this simple way. Two vectors might be related via a matrix multiplication: $\mathbf{V} = \mathbf{M}\cdot\mathbf{U}$, and the question is whether this equation holds for all observers if $\mathbf{U}$ and $\mathbf{V}$ are 4-vectors. Since the vectors change under coordinate transformations, the components of the matrix $\mathbf{M}$ must also change in a special way to compensate for this in order that the equation remains valid. Such 'physical matrices' are what we mean by tensors.

One aspect of the previous general discussion will be worth keeping firmly in mind throughout, and this relates to coordinate transformations. As presented earlier, basis transformations are something that we carry out fairly arbitrarily, just as in describing Euclidean space we might at any stage choose to rotate our coordinate system. How does this relate to relativity, and in particular to SR, where the coordinate transformations of interest are the Lorentz transformation? In fact, it is important to realize that GR makes no distinction between coordinate transformations associated with motion of the observer and a simple change of variable. For example, we might decide that henceforth we will write down coordinates in the order (x, y, z, ct) rather than (ct, x, y, z) (as is indeed the case in some formalisms). GR can cope with these changes automatically. But this flexibility of the theory is something of a problem: it can sometimes be hard to see when some feature of a problem is 'real', or just an artefact of the coordinates adopted. Sometimes a distinction is made between **active Lorentz transformations** and **passive Lorentz transformations**; a more common term for the latter class is **gauge transformation**. The term gauge always refers to some freedom within a theory that has no observable consequence, e.g. the arbitrary value of $\boldsymbol{\nabla}\cdot\mathbf{A}$, where $\mathbf{A}$ is the vector potential in electrodynamics.

9.1 Transformation of vectors and tensors

Scalar fields are objects that have no index, and that do not change under a general coordinate transformation:

$$\phi'(x') = \phi(x). \qquad (158)$$

Scalars are thus also **relativistic invariants**, which are the same for all observers. An important example of this is the proper time, $\tau(x)$. But not all numbers are scalars. Number density, n, is not because of length contraction. Similarly, an energy density, ρc^2, is not a scalar, since both the volume and energy in that volume will change under a Lorentz boost.

But the components of vectors do change under coordinate transformations. There are two cases to consider:

(a) **Contravariant vectors**, V^μ (or sometimes simply **vectors**): These have an upper index and transform according to

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\, V^\nu, \qquad (159)$$

following the application of the chain rule to the 4-vector line element dx^μ:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\, dx^\nu. \qquad (160)$$

(b) **Covariant vectors**, U_μ (or sometimes simply **co-vectors**); this name has nothing to do with general covariance. These are the components of the 1-forms discussed above. They have a lower index and transform according to

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu. \tag{161}$$

By definition, vectors and co-vectors combine to make an invariant under coordinate transformations:

$$\boxed{A^\mu B_\mu = \text{invariant scalar.}} \tag{162}$$

In this process of **contraction**, the summation convention is assumed when the same index repeats once upstairs and once downstairs – and an analogous rule is used in the coordinate transformations. The above transformation laws are set up to guarantee that contraction yields an invariant, as may be seen by thinking of the transformation in matrix terms: $\mathbf{V}' = \mathbf{M} \cdot \mathbf{V}$. If the 'dot product' of a new pair of covariant and contravariant vectors is to be unchanged after transformation, we want the product of one transformation matrix times the transpose of the other to be the identity. The transformation coefficients do satisfy this, through the properties of partial derivatives:

$$\boxed{\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} = \delta^\beta_\alpha = \text{diag}(1, 1, 1, 1).} \tag{163}$$

Notice that different things are being held constant in the two partial derivatives: the x coordinates in the first one, but the x' coordinates in the second. Taking this step by step, $dx^\beta = (\partial x^\beta / \partial x'^\mu)\, dx'^\mu$ is the chain rule. Now divide by dx^α and use the fact that $\partial x^\beta / \partial x^\alpha$ gives the Kronecker delta by definition.

Example: if ϕ is a scalar field, then $\partial\phi/\partial x^\mu$ is a co-vector, since

$$d\phi = \frac{\partial \phi}{\partial x^\nu}\, dx^\nu \tag{164}$$

and $d\phi$ has to be invariant. This co-vector then transforms as

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}, \tag{165}$$

as expected. Building on this, we will commonly use the following notations for coordinate derivatives:

$$\boxed{\partial_\mu \equiv \frac{\partial}{\partial x^\mu}; \quad \phi_{,\mu} \equiv \partial_\mu \phi;} \tag{166}$$

i.e. a comma in the subscript denotes a coordinate derivative. Similarly, $\partial^\mu \equiv \partial/\partial x_\mu$, although we have yet to show how to relate coordinates x_μ to x^μ.

Equation (164) is a good example of the need for both vectors and co-vectors in generating invariants. These elements are said to be **dual** to each other, and they have equal significance. A familiar example of dual quantities is column and row vectors. Both are needed to form a number by matrix multiplication (**inner product**). Here, when we represent vectors and co-vectors in terms of components, we distinguish the two types of object by indices that are either upstairs or downstairs. When we carry out contraction to make an invariant, it is therefore essential for this to involve pairs of indices of each type: $A_\mu A^\mu$ is invariant, but $A^\mu A^\mu$ is not.

9.1.1 Tensors of arbitrary rank

Tensors have components governed by a number of indices. The **rank** of the tensor is the number of indices – so scalars are rank-0 tensors and vectors are rank-1 tensors. A tensor with upper indices $\alpha, \beta \ldots$ and lower indices $\mu, \nu, \ldots$ transforms like a product of vectors of different types $U^\alpha V^\beta \ldots W_\mu X_\nu \ldots$. e.g.,

$$\boxed{T'^\mu{}_{\alpha\nu} = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\kappa}{\partial x'^\alpha} \frac{\partial x^\rho}{\partial x'^\nu} T^\sigma{}_{\kappa\rho}.} \tag{167}$$

T can be vector-like (all indices up), co-vector-like (all down), or mixed. A good example is the Kronecker delta: we wrote this with one index of each kind, so it transforms as

$$\delta'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \delta^\alpha{}_\beta, \tag{168}$$

from which we can see that the Kronecker delta is unchanged after transformation, as is necessary.

Note that in principle the *location* of the various indices matters: a given tensor may have a number of indices, and any one of them might be up or down. Thus, for example, $X_{\alpha\beta}{}^\gamma$ is not the same as $X^\gamma{}_{\alpha\beta}$, so it would be ambiguous just to list the up and down indices in order, as $X^\gamma_{\alpha\beta}$. But where there are only two indices, and where the tensor is symmetric, it is possible to be sloppy: there is no ambiguity if we write δ^α_β rather than $\delta^\alpha{}_\beta$.

The tensor transformation law takes its given form because of the need to ensure covariance. Consider the expression $T_{\mu\nu} A^\mu B^\nu =$ scalar, which we want to hold in all frames:

$$T'_{\mu\nu} A'^\mu B'^\nu = T'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} A^\alpha \frac{\partial x'^\nu}{\partial x^\beta} B^\beta \Rightarrow T_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T'_{\mu\nu}, \tag{169}$$

where the last result is an example of the tensor transformation law. In effect, we transform each index separately, as we would if the tensor was just a product of different vectors.

These various effects of coordinate transformations gives us a powerful principle for constructing relativistically valid laws of physics: the **principle of manifest covariance**. If an equation is written in terms of tensors, and if the free indices match in type on either side, then we know that it must apply for all coordinate systems. So $A^{\alpha\beta} B_\alpha C_\beta D^\gamma = 17 E^\gamma$ is manifestly covariant (but $A^\alpha = B_\alpha$ is not). Conversely, if we are told that an equation is covariant and everything in it bar one quantity is known to be a tensor or vector, that extra quantity must also be a tensor – i.e. it must have the right transformation law. Of course, not all covariant equations are manifestly covariant, such as the geodesic equation of motion $dU^\mu/d\tau + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0$. This is covariant (we derived it from SR using the equivalence principle), but $dU^\mu/d\tau$ is not a 4-vector, and the connection $\Gamma^\mu_{\alpha\beta}$ is *not* a tensor.

9.2 The metric tensor

Now we can deal with how to change indices from upstairs to downstairs, and vice versa. This is accomplished using the **metric tensor**, which we saw was required by the Equivalence Principle in order to obtain a generally invariant spacetime interval:

$$c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu. \tag{170}$$

Earlier, we defined this via a coordinate transformation from a Local Inertial Frame with coordinates ξ^μ:

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}. \tag{171}$$

We now recognise this as an example of the general tensor transformation law. Because $\eta_{\alpha\beta}$ is symmetric, the general metric tensor must also satisfy $g_{\mu\nu} = g_{\nu\mu}$.

We saw that in SR we wanted to write the spacetime interval as $c^2 d\tau^2 = dx_\mu dx^\mu$, so this suggests that we use the existence of the metric to *define* co-vector equivalents of vectors:

$$\boxed{V_\mu \equiv g_{\mu\nu} V^\nu.} \tag{172}$$

So we always use $g_{\mu\nu}$ to lower indices, and this applies to tensors too: $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$. To go in the opposite direction, we require the inverse of the covariant metric, $g^{\mu\nu}$:

$$\boxed{g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu.} \tag{173}$$

By manifest covariance, we see that this definition proves $g^{\mu\nu}$ to be a tensor, which again is symmetric.

Note that in SR, the raising/lowering operation simply changes the sign of the spatial parts (if Cartesian x, y, z coordinates are employed). In GR the operations are more complicated (as they are in SR if e.g. spherical polar coordinates are used).

10 Parallel transport and covariant differentiation of tensors

Having set up the concept of tensors, it is natural to ask how we will carry out calculus with these objects. From our earlier discussion, we know that this is problematic in general. If a vector obeys a coordinate transformation $V'^\mu = \Lambda^\mu_\nu V^\nu$, then differentiating this equation will produce derivatives of the transformation coefficients, Λ^μ_ν, which means that the derivative of V^μ will not obey the tensor transformation law. The only exception is when Λ^μ_ν are constants, as in SR.

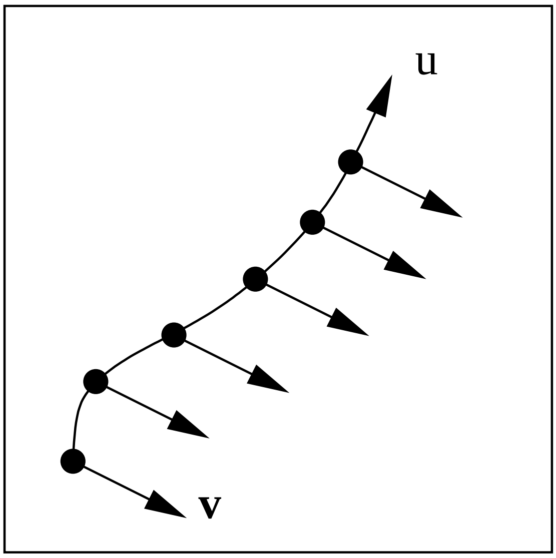

Figure 5: Parallel transport of a vector is straightforward in Euclidean space: given a vector **v** defined at some initial point, we move along a curve with tangent vector **u** to a new point, at which we create a new vector parallel to the initial **v**. In a non-Euclidean manifold, we define parallel transport as a process that applies in the tangent space (where the Euclidean approach is valid). The parallel-transported version of **v** is now not in general a vector in the manifold, and so must be projected back into the manifold. For a small step along the curve corresponding to **u**, the effect of this projection is second order in the step (think of $\cos\theta$ for small θ), but the basis vectors may change by an amount that is first order, and so the parallel-transported version of **v** will rotate in general.

To see how to evade this problem, we first introduce another. To differentiate a vector with respect to spacetime coordinates, we need to compare the value of the vector at two different points, $V^\alpha(x^\mu)$ and $V^\alpha(x^\mu + \delta x^\mu)$. But in curved spacetime it is not obvious how this comparison is to be carried out. Comparing two different vectors at the same point is fine, but somehow we need to 'copy' the vector at x^μ to $x^\mu + \delta x^\mu$. The tool for carrying this out is the process of **parallel transport**.

Imagine an observer travelling along some path, carrying with them some vector that is maintained parallel to itself as the observer moves. This is easy to imagine for small displacements, where a locally flat tangent frame can be used to apply the Euclidean concept of parallelism without difficulty (see Figure 5). But in a curved manifold, things will be more complicated. The original vector is a linear combination of basis vectors, so the question is really what happens to basis vectors under parallel transport: if they alter under this process, then the components of the transported vector at its new location will be different – i.e. it undergoes rotation after being transported. The only exception to this, as we shall see, is when we move along a geodesic and transport a vector that is tangent to the geodesic.

An example that may help clarify some of this is the surface of a sphere of radius R. The embedded 3D position vector of a point in the 2D manifold is $\mathbf{r} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, which generates basis vectors

$$\begin{aligned} \mathbf{e}_\theta &= \partial\mathbf{r}/\partial\theta = R(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \mathbf{e}_\phi &= \partial\mathbf{r}/\partial\phi = R(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0). \end{aligned} \tag{174}$$

Suppose we change θ by a small amount: the change in $\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_\phi$, and the change in $\mathbf{e}_\phi$ is similarly perpendicular to $\mathbf{e}_\theta$. So transporting these vectors along a geodesic great circle at constant ϕ does not rotate the basis. But moving along a non-great circle at constant θ will have a different effect: the dot product between the change in one basis vector and the other vector is nonzero if $\theta \neq \pi/2$. We will explore this example in more detail in the tutorials.

Without going into further detail at this point, we will now simply assume that a means of parallel transport is available: this allows us to define the **covariant derivative** of a vector $\mathbf{v}$ (the tangent vector $d/d\mu$ to a curve with parameter μ) along the curve with parameter λ (i.e. in the direction of the tangent vector $\mathbf{u} = d/d\lambda$):

$$\boxed{\boldsymbol{\nabla}_\mathbf{u}\, \mathbf{v}(\lambda) = \lim_{\epsilon\to 0} [\mathbf{v}(\lambda+\epsilon) - \mathbf{v}_\parallel(\lambda+\epsilon)]/\epsilon,} \tag{175}$$

where $\mathbf{v}_\parallel(\lambda+\epsilon)$ denotes the vector $\mathbf{v}(\lambda)$ parallel-transported to $\lambda+\epsilon$. The obvious analogous definition for scalars (which are trivial to transport) is just $\boldsymbol{\nabla}_\mathbf{u} f = df/d\lambda$. Note that the covariant derivative should not be confused with the **Lie derivative**, which is just the commutator of the two tangent vectors involved:

$$\mathcal{L}_\mathbf{u}\, \mathbf{v} = [\mathbf{u}, \mathbf{v}] = \frac{d}{d\lambda}\frac{d}{d\mu} - \frac{d}{d\mu}\frac{d}{d\lambda}. \tag{176}$$

Parallel transport clearly a reversible process: a vector can be carried a large distance and back again along the same path, and will return to its original state. However, this need not be true in the case of a loop where the observer returns to the starting point along a different path: in general, parallel transport around a loop will cause a change in a vector, and this is the intrinsic signature of a curved space.

10.1 Components of the covariant derivative

Components of the covariant derivative are introduced in terms of derivatives involving basis vectors via

$$\boxed{\boldsymbol{\nabla}_{\mathbf{e}_i}\,\mathbf{e}_j \equiv \Gamma^k_{ji}\,\mathbf{e}_k.} \tag{177}$$

For a general vector, $\mathbf{u} = u^i\mathbf{e}_i$, the effect should be linear in the components, so $\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{v} = u^i\boldsymbol{\nabla}_{\mathbf{e}_i}\mathbf{v}$. The Γ coefficients are in fact the **connection** coefficients we met earlier. Their occurrence in the context of relating vectors at two places justifies the name. Using this definition, we can show that the following definition of a geodesic is equivalent to the equation we had earlier:

$$\boxed{\boldsymbol{\nabla}_{\mathbf{v}}\,\mathbf{v} = 0;} \tag{178}$$

i.e. a geodesic curve is one that parallel transports its own tangent vector. We proceed by writing the covariant derivative in components:

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{u}}\,\mathbf{v} &= u^j\,\boldsymbol{\nabla}_{\mathbf{e}_j}v^i\mathbf{e}_i = u^jv^i\Gamma^k_{ij}\mathbf{e}_k + u^j\,\frac{\partial v^i}{\partial x^j}\mathbf{e}_i \\ &= \left(u^jv^i\Gamma^k_{ij} + u^j\,\frac{\partial u^k}{\partial x^j}\right)\mathbf{e}_k.\end{aligned} \tag{179}$$

Here, we have use the fact that covariant derivatives of scalars are the same as ordinary derivatives, since parallel transport of a scalar leaves it unchanged. So when $\mathbf{v} = \mathbf{u}$ and $\mathbf{u}$ is the 4-velocity, this is our geodesic equation, (32). To see this, note that $u^j\partial u^k/\partial x^j$ is just $du^k/d\tau$, because $u^j = dx^j/d\tau$.

If we write u^j in terms of an affine parameter as $dx^j/d\lambda$, then $\boldsymbol{\nabla}_{\mathbf{u}}\,\mathbf{v}\;d\lambda = (dv^k + \Gamma^k_{ij}v^idx^j)\,\mathbf{e}_k$. Thus in terms of components, the effect of parallel transport is to produce a change in a vector proportional both to the vector itself (rotation), and to the distance travelled:

$$\boxed{dV^\mu_{\parallel} = -\Gamma^\mu_{\alpha\beta}V^\alpha dx^\beta.} \tag{180}$$

This equation could equally well be taken as defining the components of the **affine connection**, $\Gamma^\mu_{\alpha\beta}$. The total change in going once round a small loop can be written as

$$\delta V^\mu_{\parallel} = -\oint \Gamma^\mu_{\alpha\beta}\;V^\alpha\;dx^\beta. \tag{181}$$

In general, this change does not vanish; this is illustrated in Figure 6 for transporting the basis vector $\mathbf{e}_\phi$ around a spherical triangle. Because the trajectories are all geodesics, the vector does not rotate locally – but there is a global rotation on returning to the starting point.

We are now able to construct the components of the **covariant derivative** of a vector. Differentiation involves taking the limit of $\delta V^\mu/\delta x^\nu$, but the observable change of the vector is $V^\mu(x^\nu + \delta x^\nu)$, minus the vector $V^\mu(x^\nu)$ after parallel transport to the new point. This gives the definition of the covariant derivative as

$$\boxed{\nabla_\nu V^\mu \equiv V^\mu{}_{;\nu} \equiv \partial_\nu V^\mu + \Gamma^\mu_{\alpha\nu}V^\alpha \equiv V^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu}V^\alpha,} \tag{182}$$

where we have exhibited the main notation for the covariant derivative, ∇_μ as distinct from the coordinate derivative, ∂_μ. Recall that an alternative shorthand for the latter was a comma before the index: a similar shorthand for the covariant derivative is to use a semicolon.

The covariant derivative of a tensor may be deduced by considering products of vectors, and requiring that the covariant derivative obeys the **Leibniz rule** for differentiation:

$$T^{\mu\nu}{}_{;\alpha} = (V^\mu U^\nu)_{;\alpha} = V^\mu U^\nu{}_{;\alpha} + V^\mu{}_{;\alpha}U^\nu = T^{\mu\nu}{}_{,\alpha} + \Gamma^\mu_{\beta\alpha}T^{\beta\nu} + \Gamma^\nu_{\beta\alpha}T^{\mu\beta}. \tag{183}$$

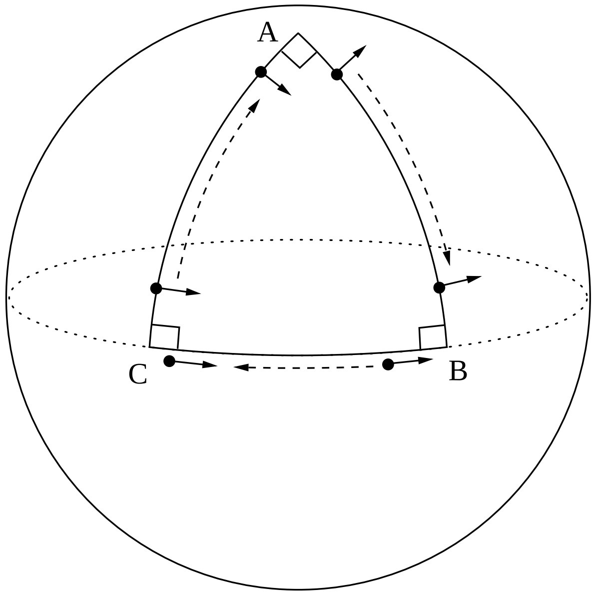

Figure 6: This figure illustrates the parallel transport of a vector around a closed loop on the surface of a sphere. For the case of the spherical triangle with all angles equal to 90°, the vector rotates by 90° in one loop. This failure of vectors to realign under parallel transport is the fundamental signature of spatial curvature, and is used to define the affine connection and the covariant derivative

This introduces a separate Γ term for each index (the appropriate sign depending on whether the index is up or down).

We can use related reasoning to get the covariant derivative of a covector. For a scalar, the covariant and coordinate derivatives are clearly equal. Consider applying this fact to a scalar constructed from two vectors:

$$\begin{aligned} &(U^\mu V_\mu)_{;\alpha} = V_\mu U^\mu{}_{;\alpha} + U^\mu V_{\mu;\alpha} = V_\mu U^\mu{}_{,\alpha} + U^\mu V_{\mu,\alpha} \\ \Rightarrow\quad & U^\mu \left(V_{\mu;\alpha} - V_{\mu,\alpha}\right) = -V_\mu \left(U^\mu_{;\alpha} - U^\mu_{,\alpha}\right) = -\Gamma^\mu_{\beta\alpha} U^\beta V_\mu = -\Gamma^\beta_{\mu\alpha} U^\mu V_\beta \end{aligned} \tag{184}$$

(swapping the labels of the two dummy indices in the last step). Since U^μ is an arbitrary vector, this allows us to deduce the covariant derivative for V_μ (different by a sign and index placement in the second term):

$$\boxed{V_{\mu;\nu} \equiv V_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} V_\alpha} \tag{185}$$

10.2 Covariant differentiation along a curve

Vectors might be defined only along a worldline, rather than everywhere – e.g. the momentum of a particle $p^\mu(\tau)$ has no meaning except on the worldline $x^\mu(\tau)$. We can project the normal derivative of a vector, A^μ, along a particle path using the 4-velocity, u^μ, so that

$$u^\lambda \partial_\lambda A^\mu = \frac{dA^\mu}{d\tau}, \tag{186}$$

which is the normal time derivative of the 4-vector with respect to the proper time of the particle. However, this time derivative is not a generally covariant quantity. We can make it a generally covariant time derivative by replacing ∂_λ by ∇_λ:

$$u^\lambda \nabla_\lambda A^\mu \equiv \frac{DA^\mu}{d\tau} = \frac{dA^\mu}{d\tau} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\lambda}{d\tau} A^\nu. \tag{187}$$

Hence if $A^\mu(\tau)$ is a tensor, then

$$\frac{DA^\mu}{d\tau} = \frac{dA^\mu}{d\tau} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu \tag{188}$$

is a tensor. Similarly, for co-vectors,

$$\frac{DB_\mu}{d\tau} = \frac{dB_\mu}{d\tau} - \Gamma^\lambda{}_{\mu\nu}\frac{dx^\nu}{d\tau}B_\lambda \tag{189}$$

is also a tensor.

10.3 The covariant derivative of the metric

To complete the logical structure of this approach, we need to relate the connection to the metric. This can be done in a number of ways, the first of which is found in the earlier discussion of the equivalence principle. In the spirit of the discussion here, a more direct route is to argue that the 'length' of a 4-vector should not be changed by the process of parallel transport. Therefore, define a vector $V_\parallel^\mu$, which is the result of parallel transport along some trajectory. Since the only change in $V_\parallel^\mu$ is that due to parallel transport, this means that its covariant derivative is zero by construction. But we want an unchanged norm: $g_{\mu\nu}V_\parallel^\mu V_\parallel^\nu = \text{constant}$.

Taking the covariant derivative of this equation and using the **Leibniz rule** for the covariant derivative (an extension of the product rule), we see that the covariant derivative of the metric must vanish:

$$\boxed{\nabla_\alpha g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \Gamma^\beta{}_{\alpha\mu}g_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu}g_{\mu\beta} = 0.} \tag{190}$$

This property of the metric is extremely convenient mathematically, since it means that the raising and lowering of indices commutes with covariant differentiation.

We have seen earlier (equation 42) that this key equation can be used to derive the relation between the connection Γ and the metric. As an alternative, it is more easily proved by working in a local inertial frame, where the first derivative of the metric and the connection vanishes. This is just the equivalence principle again, saying that inertial frames are those in which the apparent gravitational force (as expressed via derivatives of the metric) is transformed away. Saying that the covariant derivative of g vanishes is a covariant statement, which reduces to zero coordinate derivatives in a LIF. But if we prove the derivative to be zero there, then by manifest covariance it must vanish in all frames.

10.4 Gauge freedom and covariant derivatives in electromagnetism

The structure of the covariant derivative in GR is similar to something encountered in a rather different part of physics: the treatment of the electromagnetic interaction in quantum mechanics. In quantum mechanics, the phase of the wave function is unobservable: only $|\psi|^2$ matters. Therefore we can make a *global* phase transformation with no effect:

$$\psi \to e^{i\alpha}\psi. \tag{191}$$

But this makes no practical sense, as it applies everywhere: how is someone on a distant galaxy to know what value of α we chose? Clearly, this phase transformation ought to be *local*: $\alpha \to \alpha(\mathbf{x}, t)$. Such a local transformation is called a **gauge transformation**. But local phase changes mess up the Schrödinger equation. If we insert $e^{i\alpha}\psi$ for ψ into

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi = \left(\frac{\mathbf{p}^2}{2m} + V\right)\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi, \tag{192}$$

then unwanted derivatives of α are clearly going to appear. The way to fix this is to realise that α really just affects the derivatives:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + i\dot{\alpha}; \qquad \boldsymbol{\nabla} \to \boldsymbol{\nabla} + \mathbf{i}\boldsymbol{\nabla}\alpha. \tag{193}$$

We can cure this problem by adding something to the derivatives that can 'eat' the unwanted evidence of α. The **gauge-covariant derivatives** are

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + f(\mathbf{r}, t); \qquad \boldsymbol{\nabla} \to \boldsymbol{\nabla} + \mathbf{F}(\mathbf{r}, t), \tag{194}$$

where the **gauge fields** need to transform as

$$f \to f - i\dot{\alpha}; \qquad \mathbf{F} \to \mathbf{F} - i\boldsymbol{\nabla}\alpha. \tag{195}$$

But these transformations are exactly the gauge freedom of the scalar and vector potentials in electromagnetism: ϕ and $\mathbf{A}$ can be changed in this way without altering the observable $\mathbf{E}$ and $\mathbf{B}$ fields. Thus in a sense the electromagnetic interaction exists in order to allow quantum-mechanical phase to be gauge invariant. The same reasoning applies with other local symmetries of the wave function that correspond to the nuclear forces: they can only stay hidden if we introduce **gauge fields** to accomplish this. The electroweak bosons, W & Z, and the gluons of the strong force, are all gauge bosons, as is the photon.

The reasoning here is directly parallel to the gravitational case. But rather than having arbitrary local transformations of quantum-mechanical phase, we have arbitrary coordinate transformations to contend with, and we do not want these transformations to show themselves in our equations of physics. Technically, we want GR equations to obey **diffeomorphism invariance**. The cure in both cases is the same: we add a piece to the derivative that absorbs the undesired terms and makes them unobservable. Thus GR can be said to be a gauge theory of gravity.

10.5 The algorithm for generating covariant equations in GR

Covariant differentiation has two properties:

(1) It converts tensors to tensors.

(2) It reduces to ordinary differentiation in the absence of gravity ($\Gamma = 0$).

So we will satisfy the Principle of General Covariance by the following rule:

Take the equations of Special Relativity, replace $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ and all derivatives by covariant derivatives.

This procedure will allow us to take any known equation of physics in Special Relativity and convert it into a generally covariant form appropriate for working in a gravitational field. For example, we can make the fluid equation, the thermodynamic equations, Maxwell's Equations or the Dirac Equation generally covariant and study the properties of fluids, gases, photons or electrons in an arbitrary (but classical) gravitational field. In so doing, we generally appeal to the principle of **manifest covariance**: recognising a covariant equation from its form and verifying it if it reduces to SR in an inertial frame. Thus $\nabla_\mu J^\mu = 0$ must be the general conservation law, because it becomes the usual $\partial_\mu J^\mu = 0$ in flat spacetime.

This works fine – for everything except gravity, where Newtonian gravity fails even the requirements of SR: $\nabla^2\Phi = 4\pi G\rho$ is not written in terms of 4-vectors, and it contains completely anti-relativistic elements like action at a distance: changing ρ changes Φ everywhere in the universe at once, so that information is apparently propagating at the speed of light. Therefore we will need to construct a relativistic theory of gravity from scratch.

11 Spacetime curvature and gravitation

We now return to the issue of spacetime curvature, whose existence was hinted at by the existence of a metric, but where we needed an objective way to see whether or not a complicated metric might just be a rewriting of flat spacetime (as with 3D Euclidean space written in spherical polars). In some cases, we can use our intuition and experience: e.g. the surface of a sphere is a familiar and unambiguous example of a curved 2D space. But even here, care is needed, as may be seen by considering the surface of a cylinder: in an important sense, the surface of a cylinder is *not* curved, as it can be obtained from a flat plane by bending the plane without folding or distorting it. In other words, the geodesics on a cylinder are exactly those that would apply if the cylinder were unrolled to make a plane. In contrast, one must tear or fold a flat sheet to cover the surface of a sphere.

The reason that our intuition came adrift here is that we did not distinguish **extrinsic curvature** from **intrinsic curvature**. The former is concerned with the **embedding** of a surface in a higher-dimensional space, but the latter is concerned with the local properties of the surface. Although we commonly visualise curvature in terms of embedded examples such as the surface of the sphere, it is important to realise that this is not necessary, and that one can approach curved spaces directly without needing to envisage them as part of any embedding. Gauss was the first to realize that curvature can be measured without the aid of a higher-dimensional being, by making use of the intrinsic properties of a surface. For example, the curvature of a sphere can be measured by examining a (small) triangle whose sides are great circles, and using the relation

$$\text{sum of interior angles} = \pi + 4\pi \frac{\text{area of triangle}}{\text{area of sphere}}. \quad (196)$$

Very small triangles have a sum of angles equal to π, but triangles of size comparable to the radius of the sphere sample the curvature of the space, and the angular sum starts to differ from the Euclidean value.

11.1 Parallel transport and the Riemann curvature tensor

We can use parallel transport to define curvature. If vectors that are transported round a closed loop always return to their starting values, then the surface is flat. But if there is a change in the vector when we parallel transport it around a closed loop, this gives us a *geometrical definition* of curvature.

Parallel-transport a vector V^α around a small closed parallelogram with sides a^μ and b^μ, as illustrated in Figure 7. You may be concerned about whether this circuit will close, but the vectors a^μ and b^μ at either side are in practice just increments of coordinates, so they can be added in either order. From (180), the change in a vector owing to parallel transport along a displacement δx^β is

$$\delta V^\mu = -\Gamma^\mu{}_{\alpha\beta} V^\alpha \delta x^\beta. \quad (197)$$

The total change in going round the parallelogram is the difference between the value of V^μ at C when reached either by the path ABC or ADC:

$$\begin{aligned}\delta V^\mu = & -\Gamma^\mu{}_{\alpha\beta}(x) V^\alpha(x) a^\beta - \Gamma^\mu{}_{\alpha\beta}(x+a) V_p^\alpha(x+a) b^\beta \\ & +\Gamma^\mu{}_{\alpha\beta}(x) V^\alpha(x) b^\beta + \Gamma^\mu{}_{\alpha\beta}(x+b) V_p^\alpha(x+b) a^\beta. \end{aligned} \quad (198)$$

Here, we have written a subscript on $V_p^\alpha(x+a)$ etc. to emphasise that we need not the value of some general vector field $V^\alpha(x)$, but the value of the vector at x produced by parallel transport

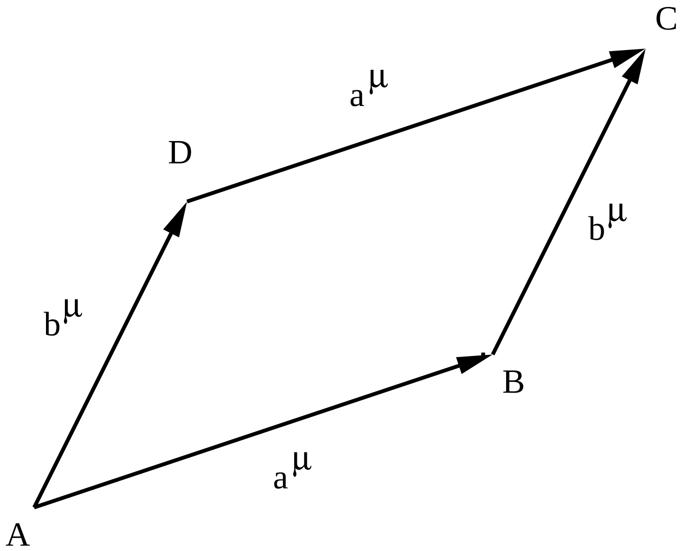

Figure 7: Consider parallel transport of a vector around the anticlockwise loop ABCD. It suffices to consider a parallelogram, as any general loop can be made up from a collection of such elements. The overall change in the vector is the change over the path ABC minus the change over the path ADC.

starting from point A. For small displacements, it is tempting to write this change in V^μ in the form of a first-order Taylor series:

$$\delta V^\mu = \frac{\partial (\Gamma^\mu{}_{\nu\beta} V_p^\nu)}{\partial x^\alpha} b^\alpha a^\beta - \frac{\partial (\Gamma^\mu{}_{\nu\beta} V_p^\nu)}{\partial x^\alpha} a^\alpha b^\beta. \tag{199}$$

The subtlety in doing so is that the action of the coordinate derivative on Γ is to differentiate a field, but the action of the derivative on V_p creates the effect of parallel transport:

$$\partial_\beta V_p^\mu = -\Gamma^\mu{}_{\alpha\beta} V_p^\alpha. \tag{200}$$

Swapping the dummy indices $\alpha \leftrightarrow \beta$ in the second term above, so there is a common factor $a^\beta b^\alpha$, and differentiating the products yields

$$\delta V^\mu = \left(\partial_\alpha \Gamma^\mu{}_{\beta\nu} V_p^\nu + \Gamma^\mu{}_{\beta\nu} \partial_\alpha V_p^\nu - \partial_\beta \Gamma^\mu{}_{\alpha\nu} V_p^\nu - \Gamma^\mu{}_{\alpha\nu} \partial_\beta V_p^\nu\right) a^\beta b^\alpha. \tag{201}$$

Now we can interpret the derivatives of V_p as parallel transport, yielding

$$\delta V^\mu = \left(\partial_\alpha \Gamma^\mu{}_{\beta\nu} V^\nu - \Gamma^\mu{}_{\beta\nu} \Gamma^\nu{}_{\gamma\alpha} V^\gamma - \partial_\beta \Gamma^\mu{}_{\alpha\nu} V^\nu + \Gamma^\mu{}_{\alpha\nu} \Gamma^\nu{}_{\gamma\beta} V^\gamma\right) a^\beta b^\alpha. \tag{202}$$

Now we can drop the subscript on V_p. Finally, relabelling dummy indices to make all V terms V^γ, we get

$$\delta V^\mu \equiv R^\mu{}_{\gamma\alpha\beta} V^\gamma a^\beta b^\alpha, \tag{203}$$

where $R^\mu{}_{\gamma\alpha\beta}$ is the **Riemann curvature tensor**:

$$\boxed{R^\mu{}_{\gamma\alpha\beta} \equiv \partial_\alpha \Gamma^\mu{}_{\beta\gamma} - \partial_\beta \Gamma^\mu{}_{\alpha\gamma} + \Gamma^\mu{}_{\alpha\nu} \Gamma^\nu{}_{\gamma\beta} - \Gamma^\mu{}_{\beta\nu} \Gamma^\nu{}_{\gamma\alpha}\,.} \tag{204}$$

Because the connections are not tensors, it is not immediately obvious that this expression is a tensor. To prove this, a little manipulation will demonstrate that the definition of $R^\mu{}_{\gamma\alpha\beta}$ can be written in terms of a commutator of covariant derivatives:

$$\left[\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right] V_\gamma = R^\mu{}_{\gamma\alpha\beta} V_\mu. \tag{205}$$

Since everything here has a tensor character except $R^{\mu}{}_{\alpha\beta\gamma}$, manifest covariance requires that this be a tensor also. Alternatively, the Riemann tensor may be defined geometrically as follows:

$$R(\mathbf{v}, \mathbf{u}, \ , \ ) = [\boldsymbol{\nabla}_{\mathbf{v}}, \boldsymbol{\nabla}_{\mathbf{u}}] - \boldsymbol{\nabla}_{[\mathbf{v},\mathbf{u}]}, \tag{206}$$

denoting commutators by square brackets. There are two empty slots here, one for a vector, one for a 1-form; the covariant derivative $\boldsymbol{\nabla}_{\mathbf{u}}$ maps a vector onto another vector, which then needs to 'eat' a 1-form in order to make an invariant. The Riemann tensor is thus a $(1,3)$ tensor in its simplest form.

If the Riemann curvature tensor is zero, then any vector parallel-transported round the loop will not change, and the space is flat. If the Riemann tensor is non-zero, vectors do change in general, and we deduce that the space is curved. If it is zero in one frame, then it is zero in all – i.e. *all* observers agree on whether space is flat (Riemann tensor zero) or curved (at least some elements of the Riemann tensor nonzero).

11.2 Calculating the Riemann tensor

Here we consider a useful matrix-handling method for making the calculation of the Riemann tensor, with its 256 terms, more manageable. This speeds up the book-keeping, as long as you are slick with matrix multiplication. This is not mandatory, and it is always possible to proceed by brute force, listing all the non-zero connection coefficients, and assembling the Riemann tensor by hand, term by term.

Affine connection $\Gamma^{\mu}{}_{\alpha\beta}$ Recall we can construct matrices Γ_α, with components (row=μ, column=β)

$$(\Gamma_\alpha)^{\mu}{}_{\beta} \equiv \Gamma^{\mu}{}_{\alpha\beta}. \tag{207}$$

Riemann tensor Construct 16 4×4 matrices, labelled by α and β, $B_{\alpha\beta}$, with components (row=μ, column=γ)

$$(B_{\alpha\beta})^{\mu}{}_{\gamma} \equiv R^{\mu}{}_{\gamma\alpha\beta}. \tag{208}$$

Hence B are defined by the matrix equation

$$B_{\alpha\beta} = \partial_\alpha\Gamma_\beta - \partial_\beta\Gamma_\alpha + \Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha, \tag{209}$$

and are clearly antisymmetric in ρ and σ $\Rightarrow$ we need to compute only 6 B matrices. We can then read off the Riemann tensor components from the elements of $B_{\rho\sigma}$. (Note the distinction between the *labels* ρ and σ, and the rows and columns α and γ).

Symmetries of R Soon we will need to make use of the symmetry properties of the Riemann tensor. Here we present these properties, without proof. However, many can be found by considering the definition of the Riemann tensor in a Local Inertial Frame. Here, the connections vanish and we only need to consider their derivatives. After quite a bit of work, a simple expression arises for the LIF Riemann tensor in its all-covariant form:

$$2R_{\alpha\beta\gamma\delta} = g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}. \tag{210}$$

This expression leads to the following symmetry properties:

$$\begin{aligned} R^{\alpha}{}_{\mu\rho\sigma} &= -R^{\alpha}{}_{\mu\sigma\rho} \\ R^{\alpha}{}_{\rho\sigma\mu} + R^{\alpha}{}_{\sigma\mu\rho} + R^{\alpha}{}_{\mu\rho\sigma} &= 0 \\ R_{\alpha\mu\rho\sigma} &= -R_{\mu\alpha\rho\sigma} \\ R_{\alpha\mu\rho\sigma} &= R_{\rho\sigma\alpha\mu}. \end{aligned} \tag{211}$$

Because of these symmetries, the number of independent components of the Riemann tensor is less than the raw $4^4 = 256$. In N dimensions, there are actually $N^2(N^2-1)/12$ degrees of freedom: 20 for the $N = 4$ case of spacetime.

Finally, we introduce two important tensors formed from the contractions of the Riemann tensor:

$$\boxed{\begin{array}{ll} R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\beta\mu} & \textbf{Ricci Tensor} \text{ (symmetric)}, \\ R \equiv R^{\alpha}{}_{\alpha} & \textbf{Ricci scalar}. \end{array}} \tag{212}$$

The Ricci tensor will be especially important, and it is convenient to have its explicit components:

$$R_{\mu\nu} = \partial_\nu \Gamma^{\alpha}{}_{\mu\alpha} - \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu}. \tag{213}$$

11.3 Gravitational tidal fields: the geodesic deviation equation

The Riemann curvature tensor depends on the second derivatives of $g_{\mu\nu}$, and we now show that this suggests a possible link between spacetime curvature and gravitational forces.

We know we can remove gravity locally by moving to a LIF in which the metric is as close to Minkowski as possible: $g_{\mu\nu} \to \eta_{\mu\nu}$, and its first derivatives vanish, $\partial_\alpha g_{\mu\nu} = 0$, so that the affine connections Γ are zero. The gravitational field only reveals itself via non-zero *second* derivatives of g, leading to **tidal forces** in which neighbouring points feel different accelerations. In the Newtonian case, the acceleration is $\ddot{\mathbf{x}} = -\boldsymbol{\nabla}\Phi$, so the relative acceleration of two points with spatial separation $\Delta\mathbf{x}$ thus depends on the difference in $\boldsymbol{\nabla}\Phi$ at these two points. Expanding Φ in a Taylor series, we can write this to lowest order as $\Delta\ddot{x}_i = \mathcal{T}_{ij}\Delta x_j$, where $\mathcal{T}_{ij} = -\nabla_i\nabla_j\Phi$ is the Newtonian **Tidal Field**. This describes how the separation of points becomes stretched and distorted by the gravitational tidal field. For use later on, we note that the contraction of the Newtonian tidal field can be used in Poisson's equation: $-\mathcal{T}_{ii} = \nabla^2\Phi = 4\pi G\rho$.

In the case of GR, we have seen that in the Newtonian limit the metric component $g_{00}c^2/2$ plays the role of the potential. Thus we can suspect that the signature of tidal forces will arise when second derivatives of the metric are non-zero: $\partial_\alpha\partial_\beta g_{\mu\nu} \neq 0$. This is fine in a LIF, but we will want to make a covariant description of the situation, so that we can handle tidal forces as seen by any observer. We might think about generating a GR tidal tensor by replacing coordinate derivatives with covariant ones: $\nabla_\alpha\nabla_\beta g_{\mu\nu} \neq 0$, but this won't get us very far because we know that the covariant derivative of the metric vanishes. In fact, it can be proved (see section 6.2 of Weinberg 1972) that there is a unique tensor that can be constructed from the metric and its first and second derivatives that is also linear in the second derivatives – and this is none other than the Riemann curvature tensor.

However, it is not necessary to pursue this argument, and it is relatively straightforward to show directly that the Riemann tensor arises in tidal forces. We will do this by considering how a pair of nearby free particles move apart or together, i.e. by *relative deviations of neighbouring geodesics*. Consider two such geodesics, $x^\mu(\tau)$ and $x^\mu(\tau) + y^\mu(\tau)$. For arbitrary τ, we will see how the (small) separation y^μ grows. Let P be the spacetime point at $x^\mu(\tau)$.

We now employ a useful general trick that simplifies the algebra. We first work in a local inertial frame at P; then we find an equation in this frame; finally we write this equation as a tensor equation, which by manifest covariance must give the general solution. Geodesics obey:

$$\begin{aligned} \ddot{x}^\mu + \Gamma^{\mu}{}_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta &= 0, \\ \ddot{x}^\mu + \ddot{y}^\mu + \Gamma^{\mu}{}_{\alpha\beta}(x+y)(\dot{x}^\alpha + \dot{y}^\alpha)(\dot{x}^\beta + \dot{y}^\beta) &= 0. \end{aligned} \tag{214}$$

Now, in a LIF at P, $\Gamma^{\mu}{}_{\alpha\beta}(x) = 0$, simplifying the algebra. Until the very end, these equations now only hold in the LIF. Making a Taylor expansion of the second equation to first order, and subtracting the first gives

$$\ddot{y}^{\mu} + \frac{\partial \Gamma^{\mu}{}_{\alpha\beta}}{\partial x^{\nu}} \dot{x}^{\alpha} \dot{x}^{\beta} y^{\nu} = 0. \tag{215}$$

Now $d^2y^{\mu}/d\tau^2$ is not a tensor, so different observers will generally disagree on whether it is zero or not. They *will* all agree on whether the covariant relative acceleration is zero, since it is a tensor.

The covariant derivative is

$$\begin{aligned} \frac{D^2 y^{\mu}}{d\tau^2} &= \frac{D}{d\tau}\left(\frac{Dy^{\mu}}{d\tau}\right) = \frac{d}{d\tau}\left[\frac{dy^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha} y^{\beta}\right] \text{ (since } \Gamma = 0) \\ &= \frac{d^2 y^{\mu}}{d\tau^2} + \frac{\partial \Gamma^{\mu}{}_{\alpha\beta}}{\partial x^{\nu}} \dot{x}^{\nu} \dot{x}^{\alpha} y^{\beta}. \end{aligned} \tag{216}$$

Using (215) and some index relabelling, this simplifies to

$$\frac{D^2 y^{\mu}}{d\tau^2} = \left(\frac{\partial \Gamma^{\mu}{}_{\alpha\nu}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\mu}{}_{\alpha\beta}}{\partial x^{\nu}}\right) \dot{x}^{\alpha} \dot{x}^{\beta} y^{\nu}. \tag{217}$$

Now, the term in brackets is not a tensor, but we see from comparison with the definition of the Riemann curvature tensor (equation 204), that in the LIF, the two are equal. Hence in this frame we can write

$$\boxed{\frac{D^2 y^{\mu}}{d\tau^2} = \left(R^{\mu}{}_{\alpha\beta\nu}\, \dot{x}^{\alpha} \dot{x}^{\beta}\right) y^{\nu}.} \tag{218}$$

This is the **Geodesic Deviation** equation, which describes how two adjacent particles in a gravitational field move relative to each other. It is the covariant generalisation of the Newtonian tidal equation.

As this is a tensor relation, it is valid in all frames. Importantly, it establishes the connection between curvature ($R^{\mu}{}_{\alpha\beta\nu}$) and gravity (through $D^2y^{\mu}/d\tau^2$). The meaning of the equation is as follows. In flat space, the Riemann tensor is zero, and we can use Cartesian coordinates. The solution is that y^{μ} grows (or reduces) linearly with τ (since $\Gamma = 0$ and the covariant derivative is just $\ddot{y}^{\mu}$, which is zero). If the two paths are parallel initially, then they stay parallel. In curved spacetime, though, geodesics that start off parallel may not remain so, because of the non-zero RHS. As a 2D example, two close parallel paths heading North from the equator will meet at the North Pole.

12 Einstein's field equations

In Newtonian gravity, there is a **field equation** that relates the gravitational potential, Φ, to the matter density, ρ. This is **Poisson's equation**:

$$\nabla^2 \Phi = 4\pi G\rho. \tag{219}$$

From a relativistic viewpoint, this equation is unsatisfactory in a number of ways. Most obviously, it violates causality: if ρ changes with time the gravitational potential alters instantly throughout the universe, so that information propagates faster than light. The same problem exists in electrostatics, and by this analogy we may expect that in the time-dependent situation ∇^2 should be replaced by the wave operator $-\Box = -\partial_{\mu}\partial^{\mu}$:

$$\partial_{\mu}\partial^{\mu}\Phi = -4\pi G\rho. \tag{220}$$

This looks like a good step towards curing the more general problem with Poisson's equation, which is that it is not covariant. The above equation would solve this if we replace the coordinate derivatives by covariant ones, and if Φ and ρ are invariant scalars.

But once again the electromagnetic analogy shows us that this will not be the case. In electromagnetism charge is indeed an invariant, but the charge density is affected by coordinate transformations because these alter volume elements. The relativistic electromagnetic field equations actually involve four potentials, which are the components of the 4-potential $A^\mu = (\phi/c, \mathbf{A})$:

$$\partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu, \tag{221}$$

where J^μ is the 4-current that contains the charge density and the current density: $J^\mu = (\rho c, \mathbf{j})$. The 4-current is needed in order to satisfy charge conservation via the **continuity equation**:

$$\partial_\mu J^\mu = 0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}. \tag{222}$$

We may suspect that a covariant field equation for gravity will probably be of a similar form: second derivatives of some fields are proportional to some measure of the density and flow of the matter field. But we cannot expect this equation to involve 4-vectors in the way that electromagnetism does. First of all, we have already seen that the metric tensor generates the effective gravitational forces, so we are probably looking for some tensor potential, $\Phi^{\mu\nu}$. Secondly, there will not be the analogue of a 4-current for mass. The electromagnetic 4-current governs the single conserved quantity of charge, but for matter there are *four* conserved quantities: energy and the three components of momentum. We will shortly see that expressing these conservation laws requires a tensor.

But rather than trying to go directly to the final gravitational field equation, we follow the path trodden by Einstein and start with the case of the vacuum gravitational field – i.e. we are looking for an equation of the form Tensor = 0, where the tensor contains second derivatives of the metric. We will now see that it is natural for this tensor to be related to the Riemann tensor that describes the curvature of spacetime.

12.1 Einstein equations in empty space

Our task is to generalise Newtonian gravity in empty space, for which the field equation is **Laplace's equation**: $\nabla^2 \Phi = 0$. This can be written as

$$\mathcal{T}^{\rm N}_{ii} = 0, \tag{223}$$

where

$$\mathcal{T}^{\rm N}_{ij} = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j}, \tag{224}$$

(for $i, j = 1, 2, 3$) is the 3D Newtonian **tidal tensor**, defined by

$$\Delta \ddot{x}_i = \mathcal{T}^{\rm N}_{ij}\, \Delta x_j, \tag{225}$$

which gives the tidal acceleration of neighbouring particles in a gravitational field.

We have seen that a covariant generalisation of this in GR is equation (218), the geodesic deviation equation for the tidal acceleration between two particles separated in spacetime by y^μ:

$$\begin{aligned} \frac{D^2 y^\mu}{d\tau^2} &= R^\mu{}_{\alpha\beta\nu} \dot{x}^\alpha \dot{x}^\beta y^\nu \\ &\equiv \mathcal{T}^\mu{}_\nu\, y^\nu, \end{aligned} \tag{226}$$

where $\mathcal{T}^\mu_{\ \nu} = R^\mu_{\ \alpha\beta\nu}\dot{x}^\alpha\dot{x}^\beta$ is the generally covariant **tidal tensor**. A covariant generalisation of Laplace's equation is then

$$\mathcal{T}^\mu_{\ \mu} = 0 \qquad \text{in empty space.} \tag{227}$$

In other words

$$R^\mu_{\ \alpha\beta\mu}\dot{x}^\alpha\dot{x}^\beta = 0, \tag{228}$$

or

$$R_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0, \tag{229}$$

where $R_{\alpha\beta} = R^\mu_{\ \alpha\beta\mu}$ is the Ricci tensor. For this to be true for all $\dot{x}^\alpha$, we need

$$\boxed{R_{\alpha\beta} = 0 \qquad \text{in empty space.}} \tag{230}$$

This gives us 10 **Einstein Equations** for the gravitational field of empty space.

We have thus finally seen the need to introduce spacetime curvature into our description of gravity: it has arisen because the equation of geodesic deviation provides a ready-made natural relativistic generalization of the second derivatives of the Newtonian potential. As stated, this falls short of a proof: we may wonder if there are other tensors that might play the same role. As mentioned, it can be shown (see p.133 of Weinberg 1972) that the Riemann tensor is the unique choice for a tensor that is *linear* in second derivatives of the metric. But even without this proof, we clearly have strong motivation to try to base the relativistic theory of gravity on this tensor.

12.2 The source of gravity: the energy-momentum tensor

We now need to take the last step of including the effects of matter. As stated above, the matter density is not a scalar invariant: if ρ_0 is the density of nonrelativistic matter in the rest frame of a fluid, then the density measured by an observer moving with Lorentz factor γ will be

$$\rho = \gamma^2\rho_0, \tag{231}$$

where one factor comes from Lorentz contraction, and another from the relativistic increase in mass of the fluid particles. This can be contrasted with a single factor of γ in the case of charge density, because charge *is* an invariant. As we saw earlier, a single conserved quantity obeys the continuity equation in the form of a vanishing 4-divergence of the 4-current: $\partial_\mu J^\mu = 0$. But in dynamics we have four quantities to conserve, which are the four components of the 4-momentum. Informally, what is needed is a way of writing four conservation laws for each component of P^μ. We can clearly write four equations of the above type in matrix form:

$$\boxed{\partial_\nu T^{\mu\nu} = 0.} \tag{232}$$

Now, if this equation is to be covariant, $T^{\mu\nu}$ must be a tensor: it is known as the **energy–momentum tensor** (or sometimes as the **stress–energy tensor**). The meanings of its components in words are: $T^{00} = c^2 \times$ (mass density) = energy density; T^{12} = y-component of current of x-momentum *etc.* From these definitions, the tensor is readily seen to be symmetric. Both momentum density and energy flux density are the product of a mass density and a net velocity, so $T^{0\mu} = T^{\mu 0}$. The spatial stress tensor T^{ij} is also symmetric because any small volume element would otherwise suffer infinite angular acceleration: any asymmetric stress acting on a cube of side L gives a couple $\propto L^3$, whereas the moment of inertia is $\propto L^5$.

For example, a cold fluid with density ρ_0 in its rest frame only has one non-zero component for the energy–momentum tensor: $T^{00} = c^2\rho_0$. We can write this in covariant form as

$$T^{\mu\nu} = \rho_0 U^\mu U^\nu, \tag{233}$$

where $U^\mu = \gamma(c, \mathbf{u})$ is the 4-velocity – and from now on we drop the 0 subscript, so that ρ refers to the rest-frame density. From this expression, we immediately see that indeed the density transforms as γ^2 in SR. The equations of energy and momentum conservation are contained in the 4-divergence of this tensor: $\partial_\nu T^{\mu\nu} = 0$. In the limit $\gamma \to 1$, $\mu = 0$ unpacks to the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0, \tag{234}$$

where here $\boldsymbol{\nabla}$ is the usual spatial gradient operator; the spatial components give

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla_k(\rho u_i u_k) = 0. \tag{235}$$

Together with the continuity equation, this can be manipulated into Euler's equation $du_i/dt = (\partial u_i/\partial t) + (\mathbf{u}\cdot\boldsymbol{\nabla})u_i = 0$ for a pressure-free fluid.

For an ideal fluid with pressure, in the rest frame we must have $T^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$ where p is the pressure (flux density of x momentum in the x direction etc.). A manifestly covariant SR form that reduces to this is

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu - p\,\eta^{\mu\nu}.} \tag{236}$$

For GR, we should replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$, and in the conservation law we should replace ∂_μ with the covariant derivative ∇_μ:

$$\boxed{\nabla_\nu T^{\mu\nu} = 0,} \tag{237}$$

which may be written explicitly as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu{}_{\alpha\nu} T^{\mu\alpha} = 0. \tag{238}$$

12.3 Einstein field equations with matter

We have generalised the source term from the density, ρ, to the stress-energy tensor, $T^{\mu\nu}$. Now we have to generalise $\nabla^2\Phi$. We have a second-rank tensor for the source term, so we seek a second-rank tensor involving derivatives of $g_{\mu\nu}$, which will ideally involve the curvature of spacetime. Given we have argued the empty space generalisation of the Laplace equation is $R_{\alpha\beta} = 0$, an obvious candidate to generalise the Poisson equation is to make the stress energy tensor the source of the Ricci tensor:

$$R^{\alpha\beta} = \text{constant } T^{\alpha\beta} \qquad \text{(wrong)}, \tag{239}$$

but this fails because

$$\nabla_\beta R^{\alpha\beta} = g^{\alpha\beta} \nabla_\beta R/2 \neq 0, \tag{240}$$

while conservation of the stress-energy tensor means $\nabla_\beta T^{\alpha\beta} = 0$. The important result for $\nabla_\beta R^{\alpha\beta}$ is known as the **contracted Bianchi identity**, and it will be proved in a Tutorial. With this identity, we can construct a tensor whose covariant divergence is always zero: it is the **Einstein tensor**, defined as

$$\boxed{G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R.} \tag{241}$$

The simplest consistent alternative to the failed relation given by equation (239) is then

$$G^{\mu\nu} = a T^{\mu\nu}, \tag{242}$$

where a is a constant. Solving for the constant using the Newtonian limit (next section) gives

$$\boxed{G^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}.} \tag{243}$$

This deceptively simple looking equation was written down by Einstein in 1915, marking the end of a decade of heroic struggle to incorporate gravitation into the framework of relativity as set out by him in 1905.

12.3.1 Sign conventions

Regrettably, much of the above varies from book to book; in a situation that makes the difference between cgs and SI electromagnetism seem like paradise, there are few universal conventions in GR. The distinctions that exist were analysed into three signs by Misner, Thorne & Wheeler (1973):

$$\begin{aligned} \eta^{\mu\nu} &= [S1] \times \text{diag}\,(-1,+1,+1,+1) \\ R^\mu{}_{\alpha\beta\gamma} &= [S2] \times \left(\Gamma^\mu_{\alpha\gamma,\beta} - \Gamma^\mu_{\alpha\beta,\gamma} + \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\gamma\alpha} - \Gamma^\mu_{\nu\gamma}\Gamma^\nu_{\beta\alpha}\right) \\ G_{\mu\nu} &= [S3] \times \frac{8\pi G}{c^4}\, T_{\mu\nu}. \end{aligned} \tag{244}$$

The third sign above is related to the choice of convention for the Ricci tensor:

$$R_{\mu\nu} = [S2] \times [S3] \times R^\alpha_{\mu\alpha\nu}. \tag{245}$$

With these definitions, Misner, Thorne & Wheeler (unsurprisingly) classify themselves as $(+++)$; Weinberg (1972) is $(+--)$; Hobson, Efstathiou & Lasenby (2006) and Ohanian & Ruffini (2013) are $(-+-)$; Cheng is $(++-)$; d'Inverno is $(-++)$. These notes are $(-+-)$, although prior to 2020 the course used $(-++)$ and this will be reflected in past exam papers.

12.4 Determining the constant a

The constant of proportionality between the Einstein tensor and the energy–momentum tensor can be determined by taking the Newtonian limit of slow motion in a weak time-independent field. The weak-field metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{246}$$

with $|h_{\mu\nu}| \ll 1$. The energy-momentum tensor is

$$T_{00} = \rho c^2; \qquad T_{ij} \simeq 0 \; (\ll T_{00}). \tag{247}$$

To first order in h, the affine connections are

$$\begin{aligned} \Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2} g^{\mu\nu} \left\{\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right\} \\ &\simeq \frac{1}{2} \eta^{\mu\nu} \left\{\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right\}. \end{aligned} \tag{248}$$

The Riemann tensor, equation (204), is

$$R^\mu{}_{\alpha\beta\gamma} \equiv \partial_\beta \Gamma^\mu{}_{\gamma\alpha} - \partial_\gamma \Gamma^\mu{}_{\beta\alpha} + \Gamma^\mu{}_{\beta\nu}\Gamma^\nu{}_{\alpha\gamma} - \Gamma^\mu{}_{\gamma\nu}\Gamma^\nu{}_{\alpha\beta}\,. \tag{249}$$

To $O(h)$ we ignore the $\Gamma\,\Gamma$ terms, and so obtain

$$R^\mu{}_{\alpha\beta\gamma} \simeq \frac{1}{2}\eta^{\mu\nu} \left\{\partial_\alpha\partial_\beta h_{\gamma\nu} - \partial_\nu\partial_\beta h_{\gamma\alpha} - \partial_\alpha\partial_\gamma h_{\beta\nu} + \partial_\gamma\partial_\nu h_{\alpha\beta}\right\}, \tag{250}$$

where two of the h terms cancel. The Ricci tensor is then

$$R_{\alpha\beta} = R^\mu{}_{\alpha\beta\mu} \simeq \frac{1}{2}\eta^{\mu\nu} \left\{\partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\nu\partial_\beta h_{\mu\alpha} - \partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\mu\partial_\nu h_{\alpha\beta}\right\}. \tag{251}$$

We will use only $G_{00} = R_{00} - (1/2)g_{00}R$ to work out the constant of proportionality, so we only need to determine R_{00} and R. For a static field, $\partial/\partial t = 0$, so derivatives w.r.t. β and σ vanish, leaving only the final term in the bracket:

$$\begin{aligned} R_{00} &\simeq \frac{1}{2}\eta^{\mu\nu}\left\{\partial_\mu\partial_\nu h_{00}\right\} \\ &\simeq -\frac{1}{2}\nabla^2 h_{00}, \end{aligned} \tag{252}$$

since h_{00} has no time derivative. Previously in equation (58) we found $g_{00} \simeq 1 + 2\Phi/c^2$ in this limit, so

$$R_{00} = -\frac{1}{c^2}\nabla^2\Phi, \tag{253}$$

to $O(h)$.

To get R, we note that, since $|T_{ij}| \to 0$ in the non-relativistic limit, then $|G_{ij}| \to 0$, or

$$R_{ij} - \frac{1}{2}g_{ij}R \simeq 0, \tag{254}$$

and so

$$R_{ij} \simeq \frac{1}{2}\eta_{ij}R \simeq -\frac{1}{2}\delta_{ij}R. \tag{255}$$

The Ricci scalar is then

$$R = R^\mu{}_\mu \simeq \eta^{\mu\nu}R_{\nu\mu} = R_{00} - R_{ii} = R_{00} + \frac{3}{2}R, \tag{256}$$

using equation (255) for R_{ii}. Hence $R \simeq -2R_{00}$ and $G_{00} = R_{00} - \frac{1}{2}g_{00}R \simeq 2R_{00}$. Finally the time-time component of the Einstein equations, $G_{00} = aT_{00}$, reduces to

$$-\frac{2}{c^2}\nabla^2\Phi = a\rho c^2 \tag{257}$$

Since we want this to match Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, the constant has to be $a = -8\pi G/c^4$.

A simpler way to get the same result is to contract the field equations, obtaining $R-(1/2)4R = aT$ (where $T = T^\mu_\mu$ and using $g^\mu_\mu = 4$). Hence the 2nd term in the Einstein tensor can be taken over to the RHS:

$$R^{\mu\nu} = a(T^{\mu\nu} - g^{\mu\nu}T/2). \tag{258}$$

Here we are interested in the 00 equation. For a Newtonian source, $T_{00} = \rho c^2$, and this also equals T. The equation is therefore $R_{00} = a\rho c^2/2$, which with R^{00} as given above completes the argument.

13 Cosmology

Now we have the field equations, in principle we can now solve to find the metric corresponding to a given matter content. This is hardly straightforward, because the field equations are strongly nonlinear. It is worth remarking why this is. The field equations show the response of spacetime curvature to the distribution of energy and momentum. But what causes the energy? Rest mass of particles, certainly, but we should also include energy from electromagnetic fields. Given the close analogy between gravitation and electromagnetism, we should expect that there is also an energy density associated with gravitational fields, but where is it? The vacuum field equations, $R_{\mu\nu} = 0$, have no right-hand side, so does that mean that there is no gravitational field energy, and hence no field? Not so, as we will see: for example, the vacuum field equations admit solutions corresponding to gravitational waves, which transmit energy quite happily. But this energy is not present as an explicit contribution to the energy-momentum tensor, but it is hidden within the nonlinear structure of the equations.

Given this nonlinearity, finding solutions to the field equations is hard. In most cases, progress is greatly aided by exploiting symmetry to simplify the possible form of the metric before trying to go any further. A good example of this, and the first practical application attacked by Einstein himself, is the case of cosmology: the spacetime of the universe on the largest of scales.

13.1 The cosmological constant Λ

The field equations (243) do not admit static solutions for the universe, as shown below. The first evidence for the expansion of the universe, in the sense of a tendency for all galaxy spectra to be redshifted, was published by Vesto Slipher in 1917. But Einstein was unaware of this, and in the same year produced a modification of his field equations that permitted a static model. We noted earlier that $\nabla_\nu g^{\mu\nu} = 0$, so we can add any multiple Λ of $g^{\mu\nu}$ to $G^{\mu\nu}$ and still get a quantity whose covariant divergence vanishes:

$$\boxed{G^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}.} \tag{259}$$

Λ is the **cosmological constant**: it cannot be too big, to avoid disturbing Newtonian gravity in the Solar System, but it can be important cosmologically.

What Einstein was doing here was actually addressing a problem that was known to Newton, and which he failed to solve. What does Newtonian gravity predict for a uniform infinite mass distribution? If we want it to be static on average, as Newton's intuition and experience suggested, then the gravitational force must vanish: $\Phi =$ constant. But this doesn't solve Poisson's equation. In order to have Φ and ρ both constant, we must modify the equation:

$$\nabla^2 \Phi + \lambda = 4\pi G \rho, \tag{260}$$

where $\lambda = c^2 \Lambda$ in the weak-field limit of the GR equation. Interestingly, Einstein's paper actually proposes $\nabla^2 \Phi + \lambda \Phi = 4\pi G \rho$, but this is not the correct nonrelativistic limit of his own field equation! The modified Poisson equation allows for a force that is linearly proportional to separation. In spherical symmetry, $\nabla^2 \Phi = (r^2 \Phi')'/r^2$, which integrates to give $\Phi = -\lambda r^2/3 - \alpha/r + \beta$ as the vacuum solution – creating a repulsive force proportional to r.

Einstein solved his field equations for a static model with a uniform matter density and cosmological constant, finding

$$c^2 d\tau^2 = c^2 dt^2 - (1 - r^2/R^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{261}$$

so the spatial geometry is curved, with the curvature radius given by

$$\frac{1}{R^2} = \Lambda = \frac{4\pi G \rho}{c^2}. \tag{262}$$

As we will see below, evidence was in fact already accumulating at the time that the universe was not static. But before this became clear, the picture was altered further by de Sitter, who in 1917 showed that Einstein's revised field equations admit a solution where there is only Λ, and now the matter density is zero:

$$c^2 d\tau^2 = (1 - r^2/R^2)\, c^2 dt^2 - (1 - r^2/R^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2). \tag{263}$$

In this case, the curvature is

$$\frac{1}{R^2} = \Lambda/3. \tag{264}$$

The interesting feature of de Sitter's solution is that $g_{00} \neq 1$, so that apparently there is time dilation as a function of redshift. But the fact that g_{00} is varying with radius means that there will be a gravitational acceleration field: a massless test particle placed at a given r will not stay in place. Therefore we find in the end that de Sitter space is not static, despite appearances: it was the first example of an expanding cosmological model. This will be explored further in a tutorial.

13.2 The expanding universe and the Friedmann-Robertson–Walker metric

Nonstatic cosmological spacetime in general was first fully understood in a pair of papers by Alexander Friedmann in 1922 & 1924, where he solved Einstein's equations for universes containing both matter and Λ. Subsequently, in the 1930s, Robertson and Walker independently showed that Friedmann's metric is a consequence only of symmetry arguments and so would stand even if Einstein's field equations were replaced by something more complicated. Thus the metric we are about to derive is sometimes named after Robertson & Walker, but more often after Friedmann also (the **FRW metric**).

Although the symmetry argument is general, it may help to give a little empirical motivation for the key steps. The evidence for the expansion of the universe dates back to 1913, and the first measurements of galaxy radial velocities by Vesto Slipher in Arizona. By 1917, he had accumulated 25 radial velocities, almost all of which were positive. At this point in history, therefore, it was first possible to see that the universe is expanding in the sense that all galaxies are receding from us. This is puzzling in many ways, most of all perhaps because we seem to be at the centre of the universe. That problem can be disposed of if we guess that we are part of a uniformly expanding matter distribution, in which all position vectors at time t are just scaled versions of their values at a reference time t_0, in terms of a universal time-dependent **scale factor**:

$$\mathbf{x}(t) = R(t)\mathbf{x}(t_0). \tag{265}$$

Differentiating this with respect to t gives

$$\dot{\mathbf{x}}(t) = \dot{R}(t)\mathbf{x}(t_0) = [\dot{R}(t)/R(t)]\,\mathbf{x}(t), \tag{266}$$

or a velocity proportional to the radius vector. Writing this relation for two points 1 & 2 and subtracting shows that this expansion appears the same for any choice of origin: everyone is the centre of the universe:

$$[\dot{\mathbf{x}}_2(t) - \dot{\mathbf{x}}_1(t)] = H(t)\,[\mathbf{x}_2(t) - \mathbf{x}_1(t)]; \qquad H(t) = \dot{R}(t)/R(t). \tag{267}$$

We see that an inevitable consequence of uniform expansion is **Hubble's Law**, in which recessional velocity is proportional to distance, with the constant of proportionality being $H(t)$, **Hubble's constant**. It is worth noting that this linear relation was a theoretical prediction, by Hermann Weyl in 1923. Hubble's 1929 'discovery' of an expanding universe was an attempt to test this prediction, using Slipher's redshifts.

This velocity field is **isotropic**: the same in all directions, about all points, and this isotropy is what is needed for there to be no special origin. But in order for all observers to be equivalent in this way, it is reasonable to require that *all* conditions that they experience should be the same. In particular, the density of matter should be **homogeneous**: the same at all points. In fact, it is easy to see that a matter density that is isotropic (constant at a given radius about any given point) must be homogeneous; but the opposite does not hold, and the velocity field could be anisotropic while the matter density remained uniform.

13.2.1 Cosmological time

The requirement of homogeneity is not straightforward, because the situation is non-static. An expanding matter distribution will have a density that falls with time as the scale factor increases (proportional to $1/R(t)^3$ if the matter is pressureless 'dust'). So it only makes sense to say that all observers experience the same density, if we specify that this refers to a given fixed time. But to treat this relativistically, we have to say carefully what the time coordinate is. The answer is to use the natural clocks defined by observers who are locally at rest with respect to the isotropic velocity field. These so-called **fundamental observers** inhabit local inertial frames, and so their time coordinate is the proper time ticked by freely-falling clocks. Even so, there remains the question of how to synchronise such clocks, especially if they are widely separated. The answer is slightly circular, but is consistent: we define the **cosmological time**, t, such that the density is purely a function of t. Thus in practice clocks can be synchronised at the point when the density reaches some reference value.

With this special global time coordinate (whose existence is only possible through the symmetry of the isotropic and homogeneous matter distribution), we can conclude that the metric for an isotropically expanding universe must take the following form:

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left[f^2(r)\, dr^2 + g^2(r)\, d\psi^2\right]. \qquad (268)$$

Because of spherical symmetry, the spatial part of the metric can be decomposed into a radial and a transverse part. The latter involves the length of an apparent arc on the 'sky' seen by an observer at $r = 0$; we will use a common space saving notation for this in spherical polars, where

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2. \qquad (269)$$

Distances have been decomposed into a product of a time-dependent **scale factor** $R(t)$ and a time-independent **comoving radius** r. The latter is independent of time by definition; thus a pair of events taking place at a given fundamental observer have $dr = d\psi = 0$, and we see that the cosmological time interval is indeed a proper time. The functions f and g are arbitrary; however, we can choose our radial coordinate such that either $f = 1$ or $g = r^2$, to make things look as much like Euclidean space as possible.

13.2.2 Metrics with uniform spatial curvature

Our discussion so far says that the metric separates into a part corresponding to cosmic time, and a spatial part: $c^2d\tau^2 = c^2dt^2 - d\sigma^2$. The spatial part of the metric generally corresponds to a curved space. From the assumption of homogeneity, the degree of curvature must be the same at all places, and it turns out that this symmetry requirement is enough to determine the form of the metric.

The simplest way of constructing a 3D space of constant curvature is to use our intuition with lower-dimensional spaces. A circle is a uniformly curved 1D space, embedded in 2D, and a sphere is a uniformly curved 2D space, embedded in 3D. This suggests that we embed an analogous 3D curved space in 4D, using an extra coordinate, w:

$$x^2 + y^2 + z^2 + w^2 = R^2, \qquad (270)$$

where R is the radius of curvature – which will turn out to be our scale factor. We can eliminate w as follows: write the hypersphere definition as $r^2 + w^2 = R^2$, so that $w\, dw = -r\, dr$, implying $dw^2 = r^2dr^2/(R^2 - r^2)$. The spatial part of the metric is therefore just

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + r^2dr^2/(R^2 - r^2). \qquad (271)$$

Introducing 3D polar coordinates, we have

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{272}$$

so that we get the spatial part of the metric in the form

$$d\sigma^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \tag{273}$$

Earlier, we argued that we wanted the spatial part of the metric as a factorisation between the square of the time-dependent scale factor $R^2(t)$ and functions of a time-independent comoving radius. We can achieve this here by defining a new radial coordinate $r' \equiv r/R(t)$, in which case

$$d\sigma^2 = R(t)^2 \left(\frac{dr'^2}{1 - r'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\phi^2\right). \tag{274}$$

Note that r' is a dimensionless coordinate; the dimensions are carried by the time-dependent curvature radius $R(t)$.

This may seem the end of our search for a space of constant curvature, but Friedmann (1924) had the critical intuition that this is only one of two possibilities: a space of **positive curvature**. It is possible to convert this to the metric for a space of constant **negative curvature** by the device of considering an imaginary radius of curvature, $R \to iR$. If we simultaneously let $r' \to ir'$, we obtain

$$d\sigma^2 = R(t)^2 \left(\frac{dr'^2}{1 - kr'^2} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\phi^2\right), \tag{275}$$

where the curvature index is $k = +1$ for positive curvature and $k = -1$ for negative curvature. It is also worth noting that we can take the limit $R \to \infty$, which is a **flat universe** with uncurved spatial sections (not the same as zero spacetime curvature). This can be absorbed into the above form via $k = 0$.

Henceforth we shall drop the prime on r', as it is common to denote the comoving radius by r. The full form of the FRW metric for an isotropic expanding universe is then

$$\boxed{c^2 d\tau^2 = c^2 dt^2 - R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\psi^2\right),} \tag{276}$$

where $d\psi^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$ is the 'sky angle' introduced earlier. An alternative common form of the metric is to define a different radial coordinate, χ, via $r = S_k(\chi)$ and defining the useful function

$$\boxed{S_k(\chi) = \begin{cases} \sin\chi & (k = 1) \\ \sinh\chi & (k = -1) \\ \chi & (k = 0). \end{cases}} \tag{277}$$

Then the FRW metric becomes

$$\boxed{c^2 d\tau^2 = c^2 dt^2 - R(t)^2 \left(d\chi^2 + S_k(\chi)^2 d\psi^2\right).} \tag{278}$$

Apart from the different sign of spatial curvature, there is one further critical difference between these metrics, which can be seen by considering a pair of radial trajectories with some angular difference $d\psi$, so that the transverse separation is $R\,S_k(\chi)\,d\psi$. Suppose we abandon causality and trace these paths at fixed t: for the $k = +1$ metric, we find that the transverse separation goes to zero as $\chi \to \pi$. This is the 3D analogue of great circles on the surface of a sphere, which leave the North pole and intersect again at the South pole. For $\chi = 2\pi$, the trajectories would return to their starting points. The $k = +1$ metric thus describes a **closed universe**, which like the surface of a sphere is finite in volume, but unbounded. By contrast, the $k = -1$ metric describes an **open universe** of infinite extent.

13.2.3 Distances and redshifts

It would be fair to wonder how cosmological distances are to be measured, and we can see how to do this by considering a radial geodesic: this is the path taken by a photon that reaches us from some distant galaxy. For zero proper time, we have $R(t)d\chi = c\,dt$, and hence $\chi = \int c\,dt/R(t)$. Now, this comoving distance is fixed even though time changes, and therefore the dt/R contributions at top and bottom of the integral must vanish. If these time intervals are thought of as one period of EM radiation, we see that the radiation must be redshifted:

$$\boxed{\frac{\nu_{\rm emit}}{\nu_{\rm obs}} \equiv 1+z = \frac{R(t_{\rm obs})}{R(t_{\rm emit})}.} \tag{279}$$

So the observed spectroscopic redshift tells us how much the universe has expanded since the radiation we see was emitted. It is often convenient to define a scale factor re-normalised to unity today: $a(t) = R(t)/R(t_0)$, so that $1+z = 1/a(t_{\rm emit})$.

13.3 The Einstein equations for the universe

Having found the FRW metric for homogeneous and isotropic universes, we now put this into the Einstein equations, in order to obtain the dynamical equations that the universe obeys – especially the time dependence of $R(t)$. Unfortunately, this involves a fair amount of effort. Here we sketch out the derivation; working through this in detail should be regarded as an additional tutorial exercise.

We begin with the metric, where for convenience we choose coordinates (t, r, θ, ϕ):

$$\begin{aligned} g_{\mu\nu} &= {\rm diag}\left(c^2, -R^2/\alpha, -R^2r^2, -R^2r^2\sin^2\theta\right) \\ g^{\mu\nu} &= {\rm diag}\left(1/c^2, -\alpha/R^2, -1/(R^2r^2), -1/(R^2r^2\sin^2\theta)\right). \end{aligned} \tag{280}$$

Here, R is the time-dependent scale factor; elsewhere this denotes the Ricci scalar, but we will not use that symbol here. We also use the shorthand $\alpha \equiv 1 - kr^2$. From the Euler-Lagrange equations we get the affine connections,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\beta}}{\partial x^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{281}$$

Because of the simple diagonal form of the metric, only a few of these are non-zero. For example, consider $\Gamma^0{}_{\alpha\beta}$: the factor $g^{\mu\nu}$ is only non-zero if $\nu = 0$, so of the 3 terms in brackets the first two involve derivatives of g_{00} (which all vanish) and the third involves time derivatives of $g_{\alpha\beta}$, which are only non-zero for g_{11}, g_{22} and g_{33}. So there are three non-zero $\Gamma^0{}_{\alpha\beta}$. Using similar reasoning, the complete set of non-zero elements is

$$\begin{aligned} &\Gamma^0{}_{11} = R\dot{R}/\alpha c^2; \quad \Gamma^0{}_{22} = R\dot{R}r^2/c^2; \quad \Gamma^0{}_{33} = R\dot{R}r^2\sin^2\theta/c^2; \\ &\Gamma^1{}_{01} = \dot{R}/R; \quad \Gamma^1{}_{11} = kr/\alpha; \quad \Gamma^1{}_{22} = -\alpha r; \quad \Gamma^1{}_{33} = -r\alpha\sin^2\theta; \\ &\Gamma^2{}_{02} = \dot{R}/R; \quad \Gamma^2{}_{12} = 1/r; \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta; \\ &\Gamma^3{}_{03} = \dot{R}/R; \quad \Gamma^3{}_{13} = 1/r; \quad \Gamma^3{}_{23} = \cos\theta/\sin\theta \end{aligned} \tag{282}$$

(to which should be added the symmetric counterparts $\Gamma^1{}_{10}$, $\Gamma^2{}_{20}$, $\Gamma^2{}_{21}$, $\Gamma^3{}_{30}$, $\Gamma^3{}_{31}$, $\Gamma^3{}_{32}$ – so 19 non-zero components in total: imagine doing this for a non-diagonal metric). Note that here we use dots for derivatives wrt cosmological time: $\dot{R} \equiv dR/dt$. The dot would have a different meaning if we were solving for geodesic motion in this spacetime, but the interpretation should be clear from the context.

The Ricci tensor is

$$R_{\mu\nu} = \partial_\nu \Gamma^\alpha{}_{\mu\alpha} - \partial_\alpha \Gamma^\alpha{}_{\mu\nu} + \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} - \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\mu\nu}. \tag{283}$$

Working out every component by hand would be tedious, but the effort can be minimised. First note that the Ricci tensor must be diagonal because of the form of Einstein's equations and because the metric and the energy-momentum tensor are diagonal. But we do not need all 4 diagonal components because of the spatial isotropy. It therefore suffices to consider only R_{00} and R_{11}. It may seem like we need all diagonal components because the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - R^\alpha_\alpha g_{\mu\nu}/2$ requires the trace of the Ricci tensor. But we can avoid this step by recasting the field equations through taking their trace:

$$R_{\mu\nu} - R^\alpha_\alpha g_{\mu\nu}/2 + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \;\Rightarrow\; R^\alpha_\alpha - 2R^\alpha_\alpha + 4\Lambda = -\kappa T^\alpha_\alpha, \tag{284}$$

where $\kappa \equiv 8\pi G/c^4$. So the trace of the Ricci tensor can be eliminated in terms of the trace of $T^{\mu\nu}$. It also simplifies things to write things in terms of mixed tensors, where the metric is the identity matrix. The energy-momentum tensor is $T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - pg^{\mu\nu}$, and the 4-velocity of the fluid (in comoving coordinates) is $u^\mu = (1, \mathbf{0})$, $u_\mu = (c^2, \mathbf{0})$, so $T^\mu_\nu = (\rho c^2, -p, -p, -p)$ and the trace is $T = \rho c^2 - 3p$. The Einstein equations now become

$$R^\mu_\nu = -\kappa(T^\mu_\nu - Tg^\mu_\nu/2) + \Lambda g^\mu_\nu. \tag{285}$$

Because the Ricci tensor is diagonal, $R^0_0 = g^{00}R_{00} = R_{00}/c^2$, and $R^1_1 = g^{11}R_{11} = -\alpha R_{11}/R^2$. We therefore have a pair of equations:

$$\begin{aligned} R_{00}/c^2 &= -\kappa(\rho c^2/2 + 3p/2) + \Lambda \\ -\alpha R_{11}/R^2 &= -\kappa(-\rho c^2/2 + p/2) + \Lambda. \end{aligned} \tag{286}$$

We therefore need R_{00} and R_{11}, and now there is no alternative to the work of taking the expression for Ricci in terms of connections and inserting the non-zero components. The result is

$$\begin{aligned} R_{00} &= 3\ddot{R}/R \\ R_{11} &= -(R\ddot{R} + 2\dot{R}^2 + 2c^2k)/\alpha c^2, \end{aligned} \tag{287}$$

so we have two equations that mix first and second time derivatives of $R(t)$. If we separate these out, we get a pair of **Friedmann Equations**:

$$\boxed{\begin{aligned} &\dot{R}^2 - \frac{8\pi G\rho}{3}R^2 - \frac{\Lambda}{3}c^2R^2 = -kc^2 \\ &\ddot{R} = -\frac{4\pi G}{3}(\rho + 3p/c^2)R + \frac{\Lambda}{3}c^2R. \end{aligned}} \tag{288}$$

Notice that these can be simplified to the $\Lambda = 0$ form if we define an effective density in terms of Λ (which we can do immediately at the stage of having the field equations):

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}. \tag{289}$$

This has to be accompanied by a pressure $p_\Lambda = -\rho_\Lambda c^2$.

These are the fundamental equations that govern the expansion of the Universe. The first equation has the form of an energy equation, where $\dot{R}$ is the velocity. The right-hand side is the Newtonian potential in a sphere of radius R, while the constant curvature term kc^2 serves as the total energy of the system. The second equation has the form of an acceleration, or force, equation. The first equation is more important, since the second can be obtained by differentiating the first and invoking energy conservation. Formally, this comes from the zero covariant divergence of $T^{\mu\nu}$, but the result is easy to obtain physically from a thermodynamic argument: $dE = -p\,dV$. If we say $E = \rho c^2 V$ and $V \propto R^3$, then

$$\boxed{\dot{\rho} c^2 = -3(\rho c^2 + p)\dot{R}/R.} \tag{290}$$

We can see that this equation yields ρ_Λ independent of time: Λ is indeed a cosmological constant.

Alternatively, we can treat this more fully, as an example of the matrix method for obtaining the full Riemann tensor. In this derivation, we temporarily set $c = 1$ to make the formulae shorter. To begin, we write the components of the connection and write them in matrix form, where Γ_t is the matrix with (μ, ν) component ${\Gamma^\mu}_{t\nu}$ etc.

$$\Gamma_t = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \dot{R}/R & \cdot & \cdot \\ \cdot & \cdot & \dot{R}/R & \cdot \\ \cdot & \cdot & \cdot & \dot{R}/R \end{pmatrix} \qquad \Gamma_r = \begin{pmatrix} \cdot & R\dot{R}/\alpha & \cdot & \cdot \\ \dot{R}/R & kr/\alpha & \cdot & \cdot \\ \cdot & \cdot & 1/r & \cdot \\ \cdot & \cdot & \cdot & 1/r \end{pmatrix} \tag{291}$$

$$\Gamma_\theta = \begin{pmatrix} \cdot & \cdot & \dot{R}Rr^2 & \cdot \\ \cdot & \cdot & -\alpha r & \cdot \\ \dot{R}/R & 1/r & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cot\theta \end{pmatrix} \qquad \Gamma_\phi = \begin{pmatrix} \cdot & \cdot & \cdot & R\dot{R}r^2\sin^2\theta \\ \cdot & \cdot & \cdot & -\alpha r\sin^2\theta \\ \cdot & \cdot & \cdot & -\sin\theta\cos\theta \\ \dot{R}/R & 1/r & \cot\theta & \cdot \end{pmatrix}. \tag{292}$$

The 6 independent $B_{\rho\sigma}$ matrices, ${(B_{\rho\sigma})^\alpha}_\beta \equiv {R^\alpha}_{\beta\rho\sigma}$ can be calculated from the matrix equation

$$B_{\rho\sigma} = \partial_\rho \Gamma_\sigma - \partial_\sigma \Gamma_\rho + \Gamma_\rho \Gamma_\sigma - \Gamma_\sigma \Gamma_\rho\,, \tag{293}$$

which yields

$$B_{tr} = \begin{pmatrix} \cdot & \ddot{R}R/\alpha & \cdot & \cdot \\ \ddot{R}/R & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad B_{t\theta} = \begin{pmatrix} \cdot & \cdot & \ddot{R}Rr^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \ddot{R}/R & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{294}$$

$$B_{t\phi} = \begin{pmatrix} \cdot & \cdot & \cdot & \ddot{R}Rr^2\sin^2\theta \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \ddot{R}/R & \cdot & \cdot & \cdot \end{pmatrix}; \qquad B_{r\theta} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & r^2(\dot{R}^2+k) & \cdot \\ \cdot & -(\dot{R}^2+k)/\alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{295}$$

$$B_{r\phi} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (\dot{R}^2+k)\sin^2\theta \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & -(\dot{R}^2+k)/\alpha & \cdot & \cdot \end{pmatrix} \tag{296}$$

$$B_{\theta\phi} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (\dot{R}^2+k)r^2\sin^2\theta \\ \cdot & \cdot & -(\dot{R}^2+k)r^2 & \cdot \end{pmatrix}. \tag{297}$$

To get the Ricci tensor, first find the non-zero Riemann tensor elements $R^{\alpha}{}_{\beta\rho\sigma}$ = Row α, column β of $B_{\rho\sigma}$:

$$\begin{aligned}
R^{t}{}_{rtr} &= \ddot{R}R/\alpha \\
R^{r}{}_{ttr} &= \ddot{R}/R \\
R^{t}{}_{\theta t\theta} &= \ddot{R}Rr^2 \\
R^{\theta}{}_{tt\theta} &= \ddot{R}/R \\
R^{t}{}_{\phi t\phi} &= \ddot{R}Rr^2\sin^2\theta \\
R^{\phi}{}_{tt\phi} &= \ddot{R}/R \\
R^{r}{}_{\theta r\theta} &= (\dot{R}^2+k)r^2 \\
R^{\theta}{}_{rr\theta} &= -(\dot{R}^2+k)/\alpha \\
R^{r}{}_{\phi r\phi} &= (\dot{R}^2+k)r^2\sin^2\theta \\
R^{\phi}{}_{rr\phi} &= -(\dot{R}^2+k)/\alpha \\
R^{\theta}{}_{\phi\theta\phi} &= (\dot{R}^2+k)r^2\sin^2\theta \\
R^{\phi}{}_{\theta\theta\phi} &= -(\dot{R}^2+k)r^2.
\end{aligned} \tag{298}$$

To get the Ricci tensor, $R_{\alpha\beta} = R^{\mu}{}_{\alpha\beta\mu}$, we need to add elements from the above list, possibly using the (anti-)symmetry of the Riemann tensor (from anti-symmetry of $B_{\rho\sigma}$), $R^{\alpha}{}_{\beta\rho\sigma} = -R^{\alpha}{}_{\beta\sigma\rho}$. After contracting, we find

$$R_{\mu\nu} = \begin{pmatrix} 3\ddot{R}/R & . & . & . \\ . & -A/\alpha & . & . \\ . & . & -Ar^2 & . \\ . & . & . & -Ar^2\sin^2\theta \end{pmatrix} \tag{299}$$

where $A \equiv R\ddot{R} + 2(\dot{R}^2+k)$. The Ricci scalar is then formed from contraction using the inverse metric, $R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$, to give

$$R^{\mu}_{\mu} = 6\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2+k}{R^2}\right). \tag{300}$$

The Einstein tensor is then (top-left quarter only), where powers of c are restored:

$$G^{\mu\nu} = \begin{pmatrix} -3(\dot{R}^2/c^2+k)/R^2 & . & . & . \\ . & \alpha\left[2\ddot{R}/Rc^2 + (\dot{R}^2/c^2+k)/R^2\right]/R^2 & . & . \\ . & . & \cdots & . \\ . & . & . & \cdots \end{pmatrix}. \tag{301}$$

The energy-momentum tensor is $T^{\mu\nu} = (\rho+p)u^{\mu}u^{\nu} - pg^{\mu\nu}$, and the 4-velocity of the fluid (in comoving coordinates) is $u^{\mu} = (c^2, \mathbf{0})$, so

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & . & . & . \\ . & \alpha p/R^2 & . & . \\ . & . & p/(R^2r^2) & . \\ . & . & . & p/(R^2r^2\sin^2\theta) \end{pmatrix} \tag{302}$$

and so the Einstein field equations $G^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}$ come out as before:

$$\boxed{\begin{aligned} \dot{R}^2 + kc^2 - \frac{\Lambda}{3}c^2R^2 &= \frac{8\pi G\rho}{3}R^2 \\ 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2+kc^2}{R^2} - \Lambda c^2 &= -\frac{8\pi Gp}{c^2}. \end{aligned}} \tag{303}$$

13.3.1 Cosmological dynamics

The Friedmann equation reveals the astonishing fact that there is a direct connection between the density of the universe and its global geometry. The final equation actually looks rather

Newtonian:

$$\dot{R}^2/2 - GM/R = -kc^2/2, \tag{304}$$

where $M = 4\pi G\rho R^3/3$. This is the equation of motion of a particle thrown vertically in the Earth's gravity, with $-kc^2/2$ playing the role of the total energy. But the fact that this should depend on the curvature, k, is an unexpected surprise.

For a given rate of expansion, there is a **critical density** that will yield $k = 0$, making the comoving part of the metric look Euclidean:

$$\boxed{\rho_c = \frac{3H^2}{8\pi G},} \tag{305}$$

where the **Hubble parameter** is $H \equiv R'/R$. Here, we have absorbed Λ into an effective density $\rho_\Lambda = \Lambda c^2/8\pi G$, which adds to the total. Because of this, it is common to define a dimensionless **density parameter**:

$$\boxed{\Omega \equiv \rho/\rho_c = \frac{8\pi G\rho}{3H^2}.} \tag{306}$$

It is also often convenient to put zero subscripts on these key cosmological parameters, to denote their present-day values: H_0, Ω_0. The Friedmann equation can then be rewritten in a more observational form:

$$H^2(t) - \Omega_0 H_0^2(\rho(t)/\rho_0) = -kc^2/R^2(t). \tag{307}$$

We can do two things with this. First, take the equation at $t = t_0$, which gives the curvature radius of the universe:

$$R_0 = (c/H_0)\,|\Omega_0 - 1|^{-1/2}. \tag{308}$$

As $\Omega_0 \to 1$, the scale factor diverges and the $-kc^2/2R^2$ term becomes negligible – so we can in effect introduce a 3rd solution with $k = 0$ to represent this limit. Now we can express things in terms of redshift, since

$$R(t)/R_0 \equiv a(t) = (1+z)^{-1}, \tag{309}$$

where the common notation for the dimensionless scale factor, $a(t)$, has been introduced. We will also allow for the fact that different constituents of the universe have a different dependence on scale factor and individual contributions to Ω_0. If these scale as some power of R, $R^{-\gamma}$, then the practical form of Friedmann is

$$H^2(z) = H_0^2\left[\sum_i \Omega_i(1+z)^{\gamma_i} + (1-\Omega_0)(1+z)^2\right], \tag{310}$$

where Ω_0 is the sum of all the individual Ω's. $H(z)$ is a critical quantity, since we have seen that distance is the integral $\int ct/R(t)$. Changing variables to redshift, this can be written as

$$\boxed{R_0\chi = \int \frac{c\,dz}{H(z)},} \tag{311}$$

so a knowledge of $H(z)$ gives the distance to objects in cosmology.

Thus beyond determining the curvature of the universe, the density also determines the expansion history. But to solve the Friedmann equation we will need the density as a function of time, and this evolves. Two obvious constituents of the density are pressureless nonrelativistic matter and radiation-dominated matter: these have densities that scale respectively as a^{-3} and a^{-4}, since the number density of particles is diluted by the expansion, with photons also having their energy reduced by the redshifting. Finally, we can add a time-independent **vacuum density** corresponding to the cosmological constant. Unfortunately, the Friedmann equation cannot be solved analytically if all three are present, but interesting cases arise with pairs of constituents. We restrict attention here to the $k = 0$ **flat universe**, as the curvature term in the Friedmann equation is negligible at early times. The solutions look simplest if we appreciate that normalization to the current era is arbitrary, so we can choose $a = 1$ to be at a convenient point where the densities of two main components cross over. Also, the Hubble parameter at that point (H_*) sets a characteristic time, from which we can make a dimensionless version $\tau \equiv tH_*$.

Matter and radiation Using dashes to denote $d/d(t/\tau)$, we have $a'^2 = (a^{-2} + a^{-1})/2$, which is simply integrated to yield

$$\tau = \frac{2\sqrt{2}}{3}\left(2 + (a-2)\sqrt{1+a}\right). \tag{312}$$

The limits of this expression are

$$\boxed{\begin{array}{ll} \tau \ll 1: & a = (\sqrt{2}\tau)^{1/2}. \\ \tau \gg 1: & a = (3\tau/2\sqrt{2})^{2/3}, \end{array}} \tag{313}$$

so the universe expands as $t^{1/2}$ in the radiation era, which which becomes $t^{2/3}$ once matter dominates. We therefore learn that there must be a **big bang**: a state of infinite density that lies a finite time of order $1/H_0$ in the past – of order 10 billion years. For many years the inability to say what might have happened before this was the biggest puzzle in cosmology. Today, we have a candidate theory called the **inflationary universe** that might answer this question – but it would take us too far afield to address this here.

Matter and vacuum Here, if $k = 0$, $a'^2 = (a^{-1} + a^2)/2$, which can be tackled via the substitution $y = a^{3/2}$, to yield

$$a = \left(\sinh(3\tau/2\sqrt{2})\right)^{2/3}. \tag{314}$$

This transition from the flat matter-dominated $a \propto t^{2/3}$ to exponentially expanding **de Sitter space** with $a \propto \exp(Ht)$ seems to be the one that describes our actual universe (apart from the radiation era at $z \gtrsim 10^4$).

Matter and curvature Here, a dimensionless version of the Friedmann equation is $a'^2 = a^{-1} - k$. For $k = -1$, we get

$$\tau = (a + a^2)^{1/2} - \sinh^{-1}(a^{1/2}), \tag{315}$$

which tends to undecelerated linear expansion $\tau = a$ when both are large. But for $k = +1$, the picture is very different:

$$\tau = \sin^{-1}(a^{1/2}) - (a - a^2)^{1/2}. \tag{316}$$

So a cannot exceed unity: the universe expands to this maximum value and then recollapses. Without Λ, closed models recollapse and open ones expand forever.

14 Gravitational waves

Gravitational waves represent distortions of spacetime that can self-propagate: the gravitational analogue of electromagnetic radiation. We can find them through a weak-field treatment of the metric, keeping spatial as well as time components. Note: geophysicists get upset if these are called 'gravity waves', which are deep-sea water waves.

Gravitational waves were first detected by aLIGO (advanced Laser Interferometer Gravitational Wave Observatory) in September 2015, heralding a new era in astronomical observations. This first signal came from two black holes merging, providing direct evidence for the existence of black holes themselves. A huge effort is currently going into ground and space-based detectors, along with searches for the gravitational wave signature of the Early Universe in the cosmic microwave background.

We start with a weakly perturbed gravitational field, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$, where in equation (251) we previously derived the first-order approximation to the Ricci tensor:

$$R_{\alpha\beta} = \frac{1}{2} \left\{ \partial^\mu \partial_\mu h_{\alpha\beta} - \partial_\beta \partial^\mu h_{\alpha\mu} + \partial_\beta \partial_\alpha h^\mu{}_\mu - \partial_\alpha \partial_\mu h^\mu{}_\beta \right\}, \tag{317}$$

where upper indices are raised by the inverse Minkowski metric, $\eta^{\alpha\beta}$. We can write $\partial_\beta \partial^\mu h_{\alpha\mu} = \eta^{\mu\nu} \partial_\beta \partial_\nu h_{\alpha\mu} = \partial_\beta \partial_\nu h^\nu{}_\alpha$ and defining $\Box \equiv \partial^\mu \partial_\mu$, we can write

$$\begin{aligned} R_{\alpha\beta} &= \frac{1}{2} \left\{ \Box h_{\alpha\beta} - \partial_\beta \partial_\mu h^\mu{}_\alpha + \partial_\alpha \partial_\beta h^\mu{}_\mu - \partial_\alpha \partial_\mu h^\mu{}_\beta \right\} \\ &= \frac{1}{2} \left\{ \Box h_{\alpha\beta} - \partial_\beta \omega_\alpha - \partial_\alpha \omega_\beta \right\} \end{aligned} \tag{318}$$

where

$$\omega_\alpha \equiv \partial_\mu h^\mu{}_\alpha - \frac{1}{2} \partial_\alpha h^\mu{}_\mu = \partial_\mu \left(h^\mu{}_\alpha - \frac{1}{2} \eta^\mu{}_\alpha h \right), \tag{319}$$

where h denotes the trace $h^\rho{}_\rho$. For a wave propagating through empty space, the Ricci tensor is zero, $R_{\alpha\beta} = 0$, and so the equation for the metric perturbation can be written

$$\Box h_{\alpha\beta} = \partial_\alpha \omega_\beta + \partial_\beta \omega_\alpha. \tag{320}$$

This equation would be simpler and more appealing (just the wave equation) if the terms on the RHS were absent, i.e. if $\omega_\nu = 0$. If we define a new field

$$\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h, \tag{321}$$

then the condition for making a simplified equation is

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \tag{322}$$

Why should this be so? In general it is a condition that will not hold – but we can make it so by a suitable coordinate transformation, since we have to satisfy four equations. This is an example where we exploit the **gauge freedom** in GR. Given a solution of the field equations, we can readily generate another by a change of the coordinate system: $x^\mu \to x^\mu + \epsilon^\mu$. The four free numbers represented by ϵ^μ give us the freedom to satisfy the four conditions $\partial_\mu \bar{h}^{\mu\nu} = 0$. An analogous freedom exists in electromagnetism. Maxwell's equations take the form of a relativistic wave equation, $\Box A^\mu = \mu_0 J^\mu$, but only if we insist on the **Lorenz condition**: $\partial_\mu A^\mu = 0 \Rightarrow c^2 \nabla \cdot \mathbf{A} = -\partial \phi / \partial t$.

If we choose the analogue to the Lorenz condition in gravity, $\partial_\mu \bar{h}^{\mu\nu} = 0$, then we have seen that the vacuum field equations $R_{\alpha\beta} = 0$ simplifies to $\Box \bar{h}_{\alpha\beta} = 0$. It is easy to extend this to matter, as we have $R_{\alpha\beta} = (1/2) \Box h_{\alpha\beta}$, so that $R = (1/2) \Box h$. Then $G_{\alpha\beta} = (1/2) \Box [h_{\alpha\beta} - h \eta_{h_{\alpha\beta}}/2] = (1/2) \Box \bar{h}_{\alpha\beta}$. Thus the weak-field gravitational equations take a form very similar to Maxwell's equations:

$$\boxed{\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}.} \tag{323}$$

This is a wave equation, and it shows that gravitational waves exist and propagate at the speed of light, as can be argued on very general grounds of causality. In the absence of matter, simple plane waves $\bar{h}^{\mu\nu} \propto \exp(i\mathbf{k}\cdot\mathbf{x} - kct)$ are a solution to the equation; with matter as a source of gravitation, the solution is analogous to electromagnetism:

$$\boxed{\bar{h}^{\mu\nu} = -\frac{4G}{c^4}\int \frac{[T^{\mu\nu}]}{|\mathbf{r}-\mathbf{r}'|}\, d^3r',} \tag{324}$$

in terms of the retarded source, indicated by the square brackets.

Although the gauge freedom can be exploited in the case of electromagnetism to impose $\partial_\mu A^\mu = 0$, it is well known that this does not remove the gauge freedom entirely: applying the transformation $A^\mu \to A^\mu + \partial^\mu\psi$ does not change the electric and magnetic fields, and yet any transformation that satisfies $\Box\psi = 0$ will leave the Lorenz condition unchanged. The same situation exists in GR, where the condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ is preserved under Lorentz transformations. We can therefore go further and impose the **transverse traceless gauge**. This is the gravitational analogue of the Coulomb gauge, which allows electromagnetic radiation in free space to be described in terms of a vector potential only. In gravity, the corresponding gauge definition is

$$h_{\mu 0} = 0; \quad h^\mu_\mu = 0. \tag{325}$$

This name comes from application of the wave equation and the gravitational Lorenz condition to a wave

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik^\alpha x_\alpha}, \tag{326}$$

which gives respectively

$$\begin{aligned} k^\mu k_\mu &= 0 \\ k^\mu A_{\mu\nu} &= 0. \end{aligned} \tag{327}$$

The first equation just says that the wave travels at the speed of light (k^μ is null); the second equation says that the wave is spatially transverse, since $A_{00} = 0$ in this gauge. Note that, since $h = 0$, there is no distinction in this gauge between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$.

It is interesting to look at the degrees of freedom here. When we factor out the wave behaviour, we have the 10 constant numbers in the symmetric $A^{\mu\nu}$ matrix. The transverse-traceless gauge definition supplies 5 constraints on these and the gravitational Lorenz condition supplies 3 (because the $\nu = 0$ component of $k^\mu A_{\mu\nu} = 0$ is automatically satisfied). Thus there are 2 remaining degrees of freedom: 2 polarizations of transverse waves. An alternative more general argument considers the field equations: both $G^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric, thus defining 10 equations for the metric $g^{\mu\nu}$, which also has 10 independent components. But the 10 field equations are related by the **Bianchi identity**: the 4-divergence of both the Einstein and energy–momentum tensors is zero, giving 4 constraints. We are then left with 6 degrees of freedom in the metric. But 4 of these must correspond to the gauge degree of freedom from arbitrary coordinate transformations. Thus we are left with 2 physical degrees of freedom – which are the polarization states seen in the linearized analysis.

14.1 Gravitational waves and tidal strain

Gravitational waves induce tidal forces. We previously saw the equation of geodesic deviation in the form $D^2\Delta x^\mu/d\tau^2 = \mathcal{T}^\mu_\nu \Delta x^\nu$, where the tidal tensor is $\mathcal{T}^\mu_\nu = R^\mu_{\alpha\beta\nu}\dot{x}^\alpha\dot{x}^\beta$, where $\dot{x}^\alpha$ is the observer's 4-velocity. We previously obtained an expression for the Riemann tensor in the weak-field limit (250), which can be written as

$$R_{\alpha\mu\nu\beta} = \frac{1}{2}\left(\partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\alpha\partial_\beta h_{\mu\nu}\right). \tag{328}$$

Consider a stationary observer, so that $\mathcal{T}_{\mu\nu} = c^2 R_{\mu 00\nu}$. The transverse-traceless gauge makes all terms but the first vanish, yielding

$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\ddot{h}_{\mu\nu}. \tag{329}$$

Finally, we will take the Newtonian limit of the equation of geodesic deviation by going to a LIF in which the Γ connection terms in the covariant derivative vanish, and proper time is just the time coordinate measured at a given particle. Thus the equation of motion for the relative separation of a pair of particles is

$$\Delta\ddot{x}^i = \frac{1}{2}\ddot{h}^i_j\,\Delta x^j, \tag{330}$$

and we see that the gravitational wave directly dictates a **spatial strain**, in which separations of particles are made to oscillate with an amplitude of order h. Note that the traceless nature of h_{ij} is consistent with the general requirement that the tidal tensor should be traceless (Laplace's equation).

For simplicity, choose the propagation direction to be along the z axis: $k^\mu = (\omega/c, 0, 0, \omega/c)$. Then the 0 row and column of $\mathcal{T}$ are zero through the TT gauge and the 3 row and column must be zero to satisfy transversality:

$$\mathcal{T}^\mu_\nu = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & \cdot \\ \cdot & b & -a & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \times \exp\left[\frac{i\omega}{c}\,(ct - z)\right] \tag{331}$$

(this form is determined by the requirement that $\mathcal{T}$ be traceless and symmetric). The two polarisation states correspond to $a = 0$ and $b = 0$.

Consider the effect of a passing gravitational wave on a circle of free particles at $z = 0$, first with $b = 0$. The tidal 3-acceleration on the particles is

$$\begin{aligned} \Delta\ddot{x}^i = \mathcal{T}^i_j x^j &= \begin{pmatrix} a & \cdot & \cdot \\ \cdot & -a & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \exp(i\omega t) \\ &= a \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \cos(\omega t), \end{aligned} \tag{332}$$

where we take the real part of the wave. This wave produces oscillations of the axis of a circle of free particles along the x and y axes, and is called h_+. If we write $x = x_0 + \Delta x$, then the LHS is $d^2\Delta x/dt^2$ and in the RHS we can set $x \to x_0$ to lowest order, so that $\Delta x = -(a x_0/\omega^2)\cos\omega t$ and similarly for y. Figure 8 shows the distortion. We note that this is the effect on *free particles*. If the particles are not free, any forces between them will oppose the force from the gravitational wave.

Similarly, the other polarisation, $a = 0$, gives

$$\Delta\ddot{x}^i = \mathcal{T}^i_j x^j = b \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \cos(\omega t), \tag{333}$$

i.e. independent oscillations in $x + y$ and $x - y$, producing a distortion along directions at an angle $\pi/4$ to the axes, as in Figure 9. These are called $h_\times$ waves.

The axes of the polarizations are at an angle $\pi/4$ to each other, and the wave is a simultaneous squashing and stretching, along orthogonal axes, preserving area. As noted, the effect is a *strain*:

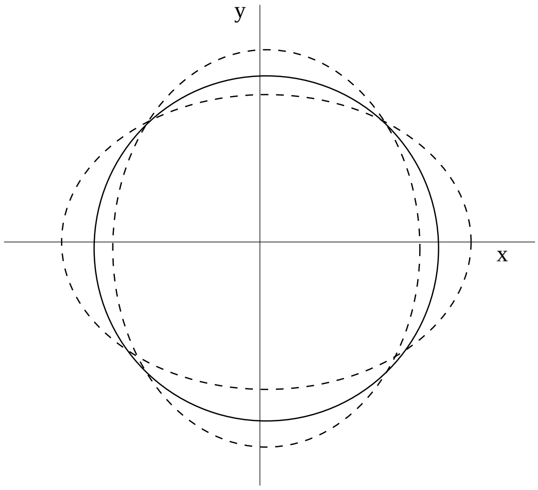

Figure 8: Effect of gravitational wave with + polarisation.

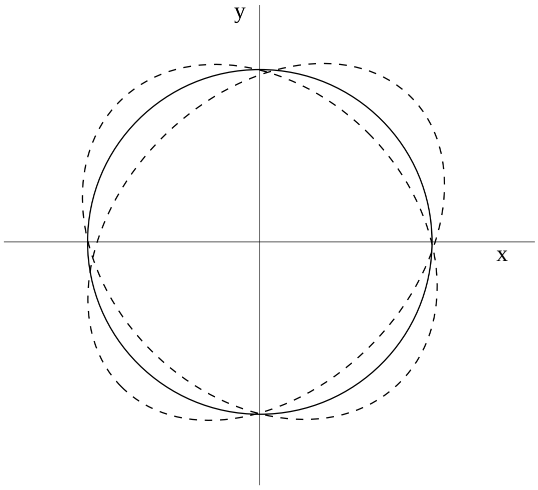

Figure 9: Effect of gravitational wave with $\times$ polarisation.

the distortion is proportional to size, motivating large (up to 4 km) detectors on Earth, e.g. aLIGO. Detectable strains are in the region $\delta x/x \sim 10^{-19} - 10^{-26}$ for the different experiments, requiring sophisticated optics. The detection of such small shifts is truly an engineering marvel, positioning a macroscopic mirror to a precision of about a billionth of the size of an atomic nucleus. This is achieved by using powerful lasers (so that Poisson errors from finite photon numbers can be reduced). Along with aLIGO, the French-Italian advanced VIRGO system came online in 2017, joining LIGO and aiding in sky localisation of gravitational-wave events. Further ground-based detectors are planned in India and China. These systems are sensitive to waves with frequencies of around 100–1000 Hz. Lower frequencies are limited by seismic noise and require the stable long baselines available in space: The European Space Agency (ESA) is developing the eLISA (evolved Laser Interferometer Space Antenna) mission, which is currently due for launch in 2034.

14.2 Energy transport by gravitational waves

Through limited lecture time, it is not possible to say too much more about gravitational waves here. But for completeness it is worth adding a few further details.. The main missing concept in the treatment so far is the transport of energy by gravitational waves, which clearly happens as the waves are able to cause the motion of matter particles: the kinetic energy must be created at the expense of field energy. There ought to exist an energy–momentum tensor that describes the effect of the waves, but where do we find it in Einstein's equations? The only explicit energy–momentum tensor is that of the matter, which vanishes for the case of vacuum radiation, so that the field equations are then just $G^{\mu\nu} = 0$. The trick that must be used is to divide the Einstein tensor into two parts, one of which represents the large-scale properties of spacetime on a scale much greater than the wavelength of the gravitational waves; the remaining part of $G^{\mu\nu}$ can be interpreted as the energy–momentum tensor of the waves. This is similar to the expansion that was made previously in obtaining the linear field equations:

$$g_{\mu\nu} = g^{(\mathrm{B})}_{\mu\nu} + h_{\mu\nu}, \tag{334}$$

where (B) denotes the background. The difference now is that the background metric is not regarded as something static, but as responding to the waves: if $G^{\mu\nu} = 0$, then the field equation for the background is

$$G^{(\mathrm{B})}_{\mu\nu} = -G^{(h)}_{\mu\nu}, \tag{335}$$

where the rhs is the contribution of the waves to the Einstein tensor. This cannot be accomplished within linearized theory, since we have previously seen that the linear Ricci tensor vanishes in the absence of matter. The Einstein tensor is then, to lowest order in h,

$$G_{\mu\nu} \simeq G^{(\mathrm{B})}_{\mu\nu} + G^{(2)}_{\mu\nu}, \tag{336}$$

where the (2) superscript denotes the second-order contribution. The effective source term for the gravitational waves is then

$$T^{\mathrm{GW}}_{\mu\nu} = \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}. \tag{337}$$

This means that it is necessary to expand the Einstein tensor to second order in h, which is an algebraic exercise that would consume too much space here. The resulting expression for the higher-order corrections to the Riemann tensor is simplified if it is averaged over several wavelengths or periods of the waves, to obtain the mean flux of energy (averaging denoted below by angle brackets), so that terms of odd order in field derivatives vanish (see chapter 35 of Misner, Thorne & Wheeler 1973). Physically, this means that the energy transport by the waves cannot be seen clearly on small scales; the same applies in electromagnetism, where the Poynting vector for a plane wave is not constant, reflecting the oscillating energy density. The final result is the **Landau-Lifshitz pseudotensor**

$$T^{\mathrm{GW}}_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \bar{h}_{\alpha\beta,\mu} \bar{h}^{\alpha\beta}{}_{,\nu} - \frac{1}{2} \bar{h}_{,\mu} \bar{h}_{,\nu} \right\rangle. \tag{338}$$

In the TT gauge, things are simpler still, because the second term inside the angle brackets vanishes. This seems to be on the right lines, as the energy density is quadratic in the field, as we would expect from the electromagnetic analogy.

We can check that this result makes sense to order of magnitude, by noting that the energy density is $U \sim (c^4/G)\dot{h}^2/c^2$, which can be compared with the electromagnetic $U \sim \epsilon_0 E^2 \sim \epsilon_0 \dot{A}^2$. We are used to guessing gravitational analogues of electromagnetism by replacing $1/4\pi\epsilon_0 \to G$, so we can see that the $1/G$ factor is plausible. The powers of c are required in order to make the units correct: c^2/G has the required dimensions of energy density.

Accepting the Landau-Lifshitz result, the above linear solutions to the wave equation can be used to obtain the energy radiated by a given configuration of matter, $T^{\mu\nu}$. The answer can be made simple in a general way first by assuming that the observer is in the far field, so that factors of $1/r$ can be taken out of integrals. Also, the fact that the source satisfies the usual conservation law can be used to recast the source term in terms of the quadrupole moment:

$$\begin{aligned} T^{\mu\nu}{}_{,\nu} &= 0 \Rightarrow T^{00}{}_{,0} = -T^{0i}{}_{,i} \\ \Rightarrow T^{00}{}_{,00} &= -T^{0i}{}_{,i0} = -(T^{i0}_{,0})_{,i} \\ \Rightarrow T^{00}{}_{,00} &= +T^{ij}{}_{,ij}, \end{aligned} \tag{339}$$

where roman letters denote spatial components, as usual. The first step uses the time element of the conservation equation, and the last step uses the spatial elements: $T^{ij}_{,j} + T^{i0}_{,0} = 0$. Now multiply $T^{ij}{}_{,ij}$ by $x^k x^\ell$ and integrate this over volume. We can use integration by parts twice in the x^i and x^j directions to obtain

$$\int T^{ij}{}_{,ij}\, x^k x^\ell \, dV = \int T^{ij}\, (x^k x^\ell)_{,ij}\, dV = 2 \int T^{k\ell}\, dV, \tag{340}$$

plus boundary terms that can be ignored because the source vanishes at infinity. Using the above result finally yields the **quadrupole formula**, which says that the field in terms of the integral over the source is related to the second time derivative of the inertia tensor:

$$h^{ij} = \frac{4G}{c^4 R} \int [T^{ij}]\, d^3x = \frac{4G}{c^4 R}\, \frac{1}{2}\, \frac{d^2}{d(ct)^2} \int [T^{00}] x^i x^j \, d^3x. \tag{341}$$

This assembles all the items needed in order to calculate the energy loss rate through gravitational radiation of an oscillating body. The linear solution gives the field in terms of $T^{\mu\nu}$ for the source, which we have just shown to be related to the time derivatives of the matter quadrupole moment. Integrating the resulting $T^{\rm GW}_{\mu\nu}$ gives the total energy loss rate. Unfortunately, the amount of algebra needed to finish the calculation is considerable; see chapter 36 of Misner, Thorne & Wheeler (1973). The final product is the **quadrupole formula**, which is the general expression for the energy lost by an oscillating nonrelativistic source via gravitational radiation:

$$\boxed{-\dot{E} = \frac{G}{5c^5} \left\langle \dddot{I}_{ij} \dddot{I}^{ij} \right\rangle}, \tag{342}$$

where the 3-tensor I^{ij} is the reduced quadrupole-moment tensor of the source

$$I_{ij} = \int \rho \, (x_i x_j - r^2 \delta_{ij}/3)\, d^3x, \tag{343}$$

$\dddot{I}$ denotes the third time derivative, and the average is over a period of oscillation. It should be no surprise that the radiation depends on the quadrupole moment; the gravitational analogue of dipole radiation is impossible owing to conservation of momentum. Whenever a mass is accelerated in astronomy, another mass or group of masses suffers an equal and opposite change in momentum, and so any attempt by an individual mass to produce dipole radiation is automatically cancelled.

An important practical application of this is for a binary of masses M_1 and M_2, orbiting with separation a around their common centre of gravity. The orbit can be assumed to be circular, because emission of gravitational radiation will damp any radial oscillations. In this case, the quadrupole formula reduces to

$$-\dot{E} = \frac{32}{5} \frac{G}{c^5}\, \mu^2 a^4 \omega^6, \tag{344}$$

where $\mu = M_1 M_2/(M_1 + M_2)$ is the reduced mass, ω is the orbital angular velocity, and Kepler tells us that $\omega^2 = G(M_1 + M_2)/a^3$. The total energy of the binary is $-GM_1M_2/2a$, and the rate of change in this must equal the power given to gravitational waves. The resulting differential equation integrates to give

$$a^4 = \frac{256 G^3}{5c^5}\, M_1 M_2 (M_1 + M_2)\, (-t); \tag{345}$$

i.e. the binary shrinks to zero size, with $a \propto (-t)^{1/4}$, with a suitable origin of time. As a consequence, the orbital frequency diverges as $\omega \propto (-t)^{-3/8}$ (although the frequency of gravitational waves will be twice this, as the quadrupole moment returns to its starting value after half an orbit). This can be expressed neatly as

$$\omega_{\rm GW} = \frac{5^{3/8}}{4} \left(\frac{G\mathcal{M}}{c^3} \right)^{5/8} (-t)^{-3/8}, \tag{346}$$

where $\mathcal{M} \equiv ([M_1 M_2]^3/[M_1 + M_2])^{1/5}$ is the **chirp mass**, which sets a characteristic timescale of the process, $G\mathcal{M}/c^3$. For $\mathcal{M} = 30M_\odot$, this time is about 0.15 ms, explaining why the massive black-hole binaries seen by LIGO involve frequencies of order kHz.

15 Black holes

Although the achievement of detecting gravitational waves is immense, the achievement of LIGO goes far beyond that, as it is able to probe the operation of gravity in the strong-field regime. The Newtonian chirp that we derived in the previous section for a binary merger will not persist to infinite frequency, as the black holes undergoing merging are objects of finite size of order the Schwarzschild radius, $2GM/c^2$. When the orbit shrinks to this size, the two black holes combine into a single more massive object, in a process that can only be calculated by numerical relativity solving Einstein's equations in full. The creation of the final object is accompanied by a **ringdown phase**, in which the black hole itself oscillates and emits radiation. The characteristic frequency of this radiation is expected to be of order the Newtonian frequency where the black holes touch, and the ringdown oscillations damp rapidly, persisting for only a few periods. Remarkably, all these predicted features were seen in the first LIGO events, taking the study of GR in one leap from weak fields to strong spacetime curvature, and showing that Einstein's theory seems to work even in this regime.

We thus now have abundant evidence that black holes exist and that the Schwarzschild metric represents a physical reality. In this last Section, we therefore return to this metric and look in more detail at some of its peculiar properties. We should note that we are studying only the simplest black holes, which are those characterised purely by mass. Two other degrees of freedom could be given to the source of the gravitational field: angular momentum (J) and charge (Q). For $Q = 0$ but $J \neq 0$, we have the **Kerr metric (1963)**; for $J = 0$ but $Q \neq 0$ we have the **Reissner–Nordstrøm metric (1916)**; the general case with all three parameters non-zero is the **Kerr-Newman metric (1965)**. Charge is less important in practice, but the Kerr solution is relevant for astrophysics, since black holes form by accretion of matter with non-zero angular momentum. Indeed, LIGO observations have been able to prove that the Kerr solution is required, and not the Schwarzschild one. The difference between the two solutions depends on the dimensionless parameter $a = Jc/GM^2$, which has a maximum allowed value of unity; the Schwarzschild solution is a good approximation if $a \ll 1$.

The Schwarzschild metric, equation (84), is given by

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{rc^2} \right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \tag{347}$$

and its detailed derivation was the subject of a tutorial. You may like to reflect on the fact that Karl Schwarzschild derived this result in 1915 – when GR was only one month old – while serving on the Russian front during the First World War; there, he had already contracted the infection that was to kill him in 1916.

In deriving the metric, we saw that it applied if the matter distribution is (a) spherical; (b) static; (c) a vacuum at the point of measurement. This general statement, that the static spherical vacuum solution of Einstein's equations is always of the Schwarzschild form, is known as **Birkhoff's theorem**. So really the above equation only applies for radii outside some minimum defined by the mass distribution. But normally we are happy to let this minimum go to zero, so that the Schwarzschild metric applies everywhere (except $r = 0$ itself, where there is a delta-function in the Einstein tensor). In this case, the metric takes the pithy name coined by Wheeler in 1967: a **Black hole**. But even for this point-mass solution, there is a characteristic length in the form of the **Schwarzschild radius**,

$$r_s \equiv \frac{2GM}{c^2}. \tag{348}$$

The behaviour of the metric is rather odd at $r = r_s$, since $g_{tt} \to 0$ and $g_{rr} \to \infty$. In fact the metric is singular at 3 'places': at $r = 0$ the potential diverges and $g_{tt} \to \infty$ and $g_{rr} \to 0$; at $r = r_s$, $g_{tt} \to 0$ and $g_{rr} \to \infty$; and at $r \to \infty$, both $g_{\theta\theta}$ and $g_{\phi\phi} \to \infty$. How should we interpret these singularities?

The behaviour as $r \to \infty$ is not a problem. Writing the Minkowski metric in spherical polars gives $c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$, and we see that two angular metric coefficients diverge as $r \to \infty$, but these singularities can be removed by using a Cartesian coordinate system. The singularity as $r \to \infty$ is thus a **coordinate singularity**. Similarly, we can create coordinate singularities in Minkowski spacetime at small r by using a change of variable: e.g. $R = r^3/3$, in which case, $dr^2 = (3R)^{-4/3} dR^2$, and the metric in terms of R becomes singular at $R = 0$). In contrast, a physical singularity would correspond to the divergence of some invariant such as the Ricci scalar $R^\mu{}_\mu$. We need to look more closely at the remaining Schwarzschild singularities to see which class they fall into.

15.1 The singularity at r_s

15.1.1 Radial trajectories of massless particles into a black hole

Consider the Lagrangian-squared for a photon travelling along a radial ($d\theta = d\phi = 0$) null ($d\tau^2 = 0$) geodesic:

$$L^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2, \tag{349}$$

where as usual a dot denotes differentiation with respect to some affine parameter, p. Using the ELII equations (71), we get

$$\begin{aligned} \frac{d}{dp}\left[\left(1 - \frac{2GM}{rc^2}\right) c\dot{t}\right] &= 0, \\ \left(1 - \frac{2GM}{rc^2}\right) \dot{t} &= k = \text{constant}. \end{aligned} \tag{350}$$

Substituting this into equation (349) for a massless particle, where $L^2 = 0$, we get

$$\dot{r}^2 = c^2 k^2. \tag{351}$$

This is the energy equation for a massless particle on a radial trajectory, and hence

$$\dot{r} = \pm c|k|, \tag{352}$$

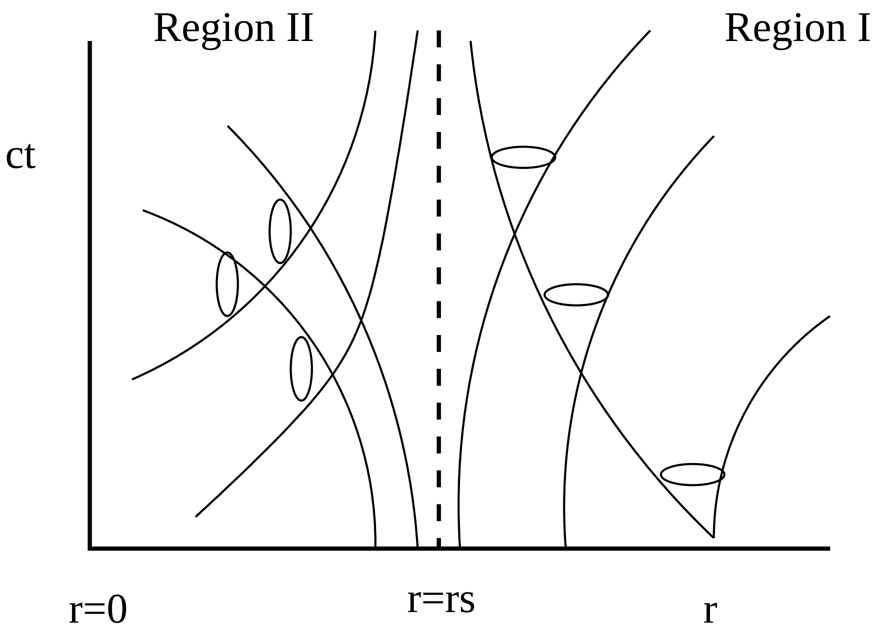

Figure 10: Geodesics and future light cones, near the Schwarzschild radius.

for incoming and outgoing particles. The velocity of the photon measured by a distant observer is

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \pm c\left(1 - \frac{r_s}{r}\right), \tag{353}$$

which has two solutions:

$$ct = \pm(r + r_s \ln|r - r_s| + \text{constant}). \tag{354}$$

For $r > r_s$, the positive solution is an outgoing photon, the negative solution ingoing.

In a spacetime diagram using the coordinates laid down by a distant observer, we can present the **light cone** structure as shown in Figure 10. The set of ingoing and outgoing geodesics (known collectively as a **null congruence**) define the boundaries of the light cones in $r - t$ space. Far from the centre, these take the 45° form of Minkowski spacetime. As we approach $r = r_s$ from larger r (region I), the light cones narrow to become parallel to the t axis. Thus in terms of *coordinate time* t, light signals in region I take an infinite time to reach r_s: information about what has fallen into a black hole remains on its surface. Conversely, in region II inside the Schwarzschild radius, the light cones tip over and point towards $r = 0$. All light signals generated at $r < r_s$ *must* move towards $r = 0$ and cannot move outwards to $r > r_s$.

Because light signals cannot escape from region II, $r = r_s$ is called an **event horizon**: it marks the boundary of events that can ever be detected in the future. In a general spacetime this can be difficult to compute because of having to consider all possible geodesics; but in the very simple metric considered here it boils down to

$$g_{rr} \to \infty. \tag{355}$$

Thus for finite time intervals as measured by a local observer at rest, $dr \to 0$ ($c^2 d\tau^2 = g_{tt} c^2 dt^2 - g_{rr} dr^2 = 0$, and $g_{tt} dt^2 = dt^2_{\text{local}}$). By symmetry, if a geodesic is unable to cross $r = 2GM/c^2$ at one location, it cannot cross anywhere, so any local condition we deduce will apply to the whole surface.

It is interesting to look at the two solutions inside r_s. For the ingoing congruence, *ct decreases* as r decreases (see diagram), which seems paradoxical. But we can define an alternative time coordinate by

$$ct' = ct + r_s \ln|r - r_s|, \tag{356}$$

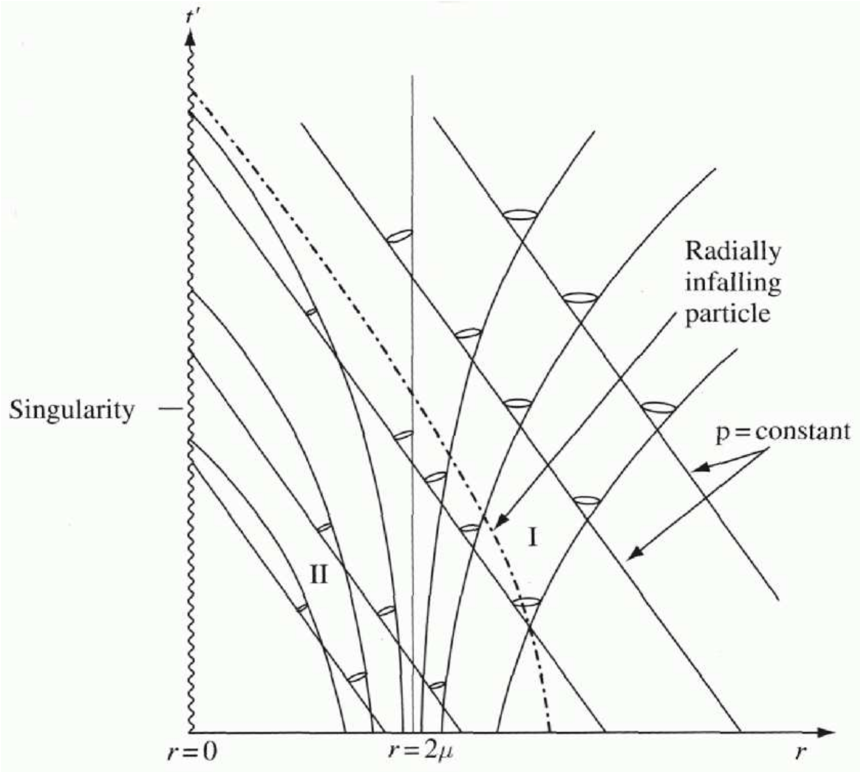

Figure 11: Geodesics and future light cones, near the Schwarzschild radius, in Advanced Eddington-Finkelstein coordinates. [Taken from Hobson, Efstathiou & Lasenby].

in terms of which the ingoing and outgoing null geodesics are

$$\begin{aligned} ct' &= -r + \text{constant} \\ ct' &= r + 2r_s \ln |r - r_s| + \text{constant}. \end{aligned} \tag{357}$$

The behaviour is perhaps clearer if we just look at the differential relation, which is the same both inside and outside the horizon:

$$\frac{cdt'}{dr} = (1 - r_s/r)^{-1}(\pm 1 + r_s/r). \tag{358}$$

At $r > r_s$ we have options of either sign, but for $r < r_s$ the term $\pm 1 + r_s/r$ is always positive, so that dt'/dr is negative for both solutions. These coordinates are called **Advanced Eddington-Finkelstein coordinates**, and are plotted in Figure 11. In this system, the light cones look more sensible, tipping over continuously as they cross the Schwarzschild radius, while the apparent singularity at $r = r_s$ disappears.

15.1.2 The infinite redshift surface

At the Schwarzschild radius the time part of the metric vanishes, so that $g_{tt} = 0$. For a stationary emitter at r, its proper time interval $d\tau$ is related to the coordinate time interval dt, by

$$c^2 d\tau^2 = g_{tt} c^2 dt^2 = c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right). \tag{359}$$

At $r = r_s$, $g_{tt} \to 0$, so $d\tau/dt \to 0$, and the ratio of emitted to observed frequency (at infinity) is

$$1 + z = \frac{dt}{d\tau} = \frac{\nu_{\text{emitted}}}{\nu_{\text{observed}}} \to \infty. \tag{360}$$

(Note: the observed dt is the same as the emitted dt – see section 4.5). $g_{tt} = 0$ is therefore an **infinite redshift** surface. This and the event horizon coincide for a Schwarzschild black hole, but not in general (they do not coincide in the case of the spinning Kerr black hole).

15.1.3 Radial trajectories of massive particles into a black hole

We can get a different perspective on the properties of the event horizon by asking what we would see if we were unfortunate enough to fall in to a black hole. For a massive particle $L^2 = c^2$, so equation (349) gives

$$\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 = c^2, \tag{361}$$

where now the dot indicates derivative w.r.t. proper time τ. The ELII equation for t gives the same constraint as equation (350), hence

$$\dot{r}^2 = c^2 k^2 - c^2 \left(1 - \frac{r_s}{r}\right). \tag{362}$$

For simplicity, let us choose $k = 1$ for a particle with zero speed at infinity, for which

$$\left(\frac{dr}{d\tau}\right)^2 = \dot{r}^2 = c^2 \frac{r_s}{r}, \tag{363}$$

which has the solution

$$c\tau = \pm \frac{2}{3} r_s \, (r/r_s)^{3/2}, \tag{364}$$

where the origin of proper time is chosen to be $\tau = 0$ at $r = 0$. Nothing strange happens on crossing the event horizon, confirming the artificial nature of the singularity there, and an observer who has fallen from infinity reaches the centre in a proper time $\tau = 2r_s/3c$ after crossing the horizon. Here, the observer meets a true singularity of divergent curvature, and will be destroyed by infinite tidal forces. There will also be tides at all points of the infalling trajectory, but if the mass is large enough (above about $10^5 \, M_\odot$) then they will not be a danger to life at $r = r_s$.

In the above equation, the minus sign corresponds to ingoing trajectories – but there is also the possibility of outgoing trajectories that are the time-reversed versions of the ingoing ones. This is quite reasonable in general, as the geodesic equation of motion is second order in time. Despite our discussion of the behaviour of light cones when analysed in coordinate time, it therefore seems that an observer *can* escape from inside the Schwarzschild radius. The behaviour in terms of coordinate time in part reflects the limitations of that coordinate. It is the time ticked by a clock at infinity, and is a reasonable description of events in region I, but not inside the horizon.

In terms of *coordinate time t*, the apparent velocity of the infalling observer is

$$v = \frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \pm c \left(\frac{r_s}{r}\right)^{1/2} \left(1 - \frac{r_s}{r}\right), \tag{365}$$

which vanishes as $r \to r_s$. Since t is the proper time of a stationary observer at infinity, as far as they are concerned, the particle *never crosses* the event horizon. In principle, we could verify that it is still there by programming it to emit a photon radially outwards at some very large time in the future; but this is not practical, since the body crosses the event horizon after a finite *proper* time. If it releases photons at anything like a constant proper rate, there will be a last photon emitted prior to crossing, and external observers will receive no further information. If we cannot receive signals that the 'hovering' body emits, can we verify its presence directly by sending a rocket to retrieve the body? The answer is no: even travelling at the speed of light, it is impossible to catch up with a falling body once it has been left to fall for a critical time. This

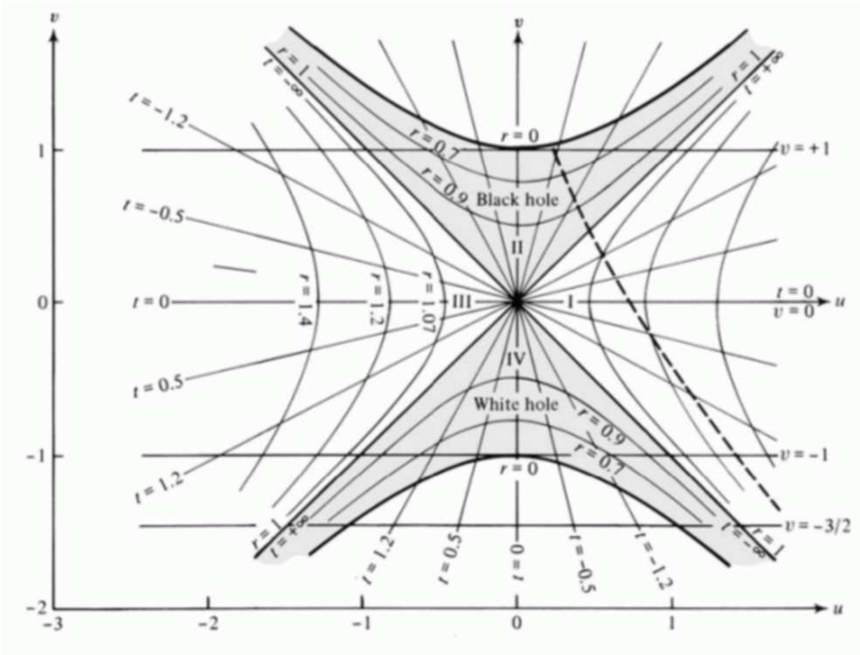

Figure 12: A diagram of the maximally extended Schwarzschild spacetime, as expressed in Kruskal-Szkeres coordinates. Light cones are all orientated at 45° in this coordinate system. Regions I & II correspond to the exterior and interior of the black hole, and light cannot escape from the latter region. But now we add the white-hole region IV, which also contains a singularity at $r = 0$, from which light *can* escape into region I. But there is also region III, which is apparently a distinct universe that is asymptotically Minkowski. [taken from Ohanian & Ruffini].

is not surprising: time dilation freezes the infall close to the horizon, but as the rescue mission approaches the horizon, this relative time dilation is lessened, unfreezing the infalling body.

These arguments show that a body undergoing gravitational collapse (such as a massive star at the end of its lifetime) will rapidly become equivalent in all practical terms to a pre-existing Schwarzschild black hole. The existence of time-reversed solutions shows that in principle this Schwarzschild metric may at some time in the future emit material, which would then be termed a **white hole**. But this is hardly a practical possibility, as it requires outgoing initial conditions to be set at $r = 0$.

15.2 Kruskal-Szekeres coordinates

We have clarified that $r = r_s$ is just a coordinate singularity for a distant observer in the Schwarzschild metric, but the discussion of ingoing and outgoing geodesics has revealed some paradoxical features depending on the time coordinate we use. To try to clarify this, we can introduce **Kruskal-Szekeres coordinates**, in which light cones have a behaviour that closely parallels that of Minkowski spacetime.

The new time and space coordinates are

$$v = \left|\frac{r}{r_s} - 1\right|^{1/2} e^{r/2r_s} \sinh\left(\frac{ct}{2r_s}\right), \quad (366)$$

$$u = \left|\frac{r}{r_s} - 1\right|^{1/2} e^{r/2r_s} \cosh\left(\frac{ct}{2r_s}\right), \quad (367)$$

for $r > r_s$, and sinh $\leftrightarrow$ cosh for $r < r_s$, in terms of which the metric becomes

$$c^2 d\tau^2 = 4r_s^2 \frac{r_s}{r} e^{-r/r_s} \left(dv^2 - du^2\right) - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{368}$$

where the time-like and space-like nature of v and u is apparent. The coordinate definitions give

$$u^2 - v^2 = (r/r_s - 1)\, e^{r/r_s}, \tag{369}$$

where lines of constant r lie on hyperbolae, and

$$\frac{v}{u} = \tanh\left(\frac{ct}{2r_s}\right), \tag{370}$$

where different time slices lie on straight lines in the v-u plane with different slopes. For a photon travelling along a radial null geodesic ($d\theta = d\phi = 0$ and $d\tau = 0$), the metric reduces to $dv^2 = du^2$ and

$$u = \pm(v + \text{const}), \tag{371}$$

showing that light cones are at 45° in these coordinates.

Inspection of the definition of the $u - v$ coordinates shows that the whole of the region $0 < r < \infty$, $-\infty < t < \infty$ is covered by the region above the diagonal line $v = -u$, which itself corresponds to $t = -\infty$. For $r > r_s$, u is positive and v ranges from $-u$ to $+u$ as t goes from $-\infty$ to $+\infty$; for $r < r_s$, v is positive and u ranges from $-v$ to $+v$ as t goes from $-\infty$ to $+\infty$. But there seems no reason not to consider the reflection of this region: r depends on $u^2 - v^2$ and t on v/u, so a double sign change would leave r and t unaltered. If we extend the spacetime to the full $u-v$ plane in this way, it splits into 4 parts separated by $r = r_s$ and $t = \pm\infty$, where v and u are well-behaved (see Figure 12). Region I, where $r > r_s$ and $-\infty < t < \infty$, is asymptotically Minkowski space, while Region II is inside the Schwarzschild radius, $r < r_s$. An observer in Region I would see the $t = -\infty$ Schwarzschild surface, but if they fell into the black hole they would cross the $t = \infty$ event horizon, which they would only see after passing through it. Region IV is the interior to a **White Hole**, which appears to occupy the same space as the black hole, but is time-reversed and bounded by $t = -\infty$. Region III is a second asymptotic Minkowski space where time appears reversed, which can be seen upon passing through the black hole event horizon, but cannot be entered from region II. Finally, the geometry of lines of constant v form an **Einstein-Rosen Bridge**, a non-traversable **wormhole**, between the two Minkowski spacetimes. This solution has received much exposure in popular literature, but it must be emphasised that the extension of the geometry is not required, and we have no reason to think that the region-III 'universe' exists, much less that it might represent a different part of our own external spacetime.

To journey further into Black Holes, see Andrew Hamilton's excellent animations and explanations at `https://jila.colorado.edu/~ajsh/insidebh`

16 Final remarks

In this introduction to General Relativity, we have developed the physical principles on which the theory is based – the Equivalence Principle, and the Principle of General Covariance. We have looked at applications to orbits, the precessing of perihelia, the bending of light in a gravitational field, and time delay, all based on the idea that spacetime has a metric structure. We have introduced Riemann's description of intrinsic spacetime curvature, and shown that the Riemann tensor provides a natural relativistic counterpart to Newtonian tidal forces, leading to the formulation of Einstein's gravitational field equations in terms of the Riemann tensor. The generation of this theory, based on a marriage of intuitive simplicity and general mathematical principles, must rank as one of the supreme achievements of human creativity.

GR has practical applications in the Global Positioning System (GPS), and satellite navigation. It has major applications in cosmology where it is needed not only for cosmological models, but also in understanding structure formation and evolution, the Early Universe, and the vacuum density as the explanation of the accelerating Universe. With the detection of gravitational waves, a new window on the Universe has now opened – which will allow us to see into the most extreme of objects and back to the earliest times. This revolution must be considered one of the most impressive achievements of experimental physics, overshadowing even the detection of the Higgs boson.

GR is not the last word in gravity, however. Alternative field equations have been proposed and generally fail to match observations any better than the simplest theory. But Einstein's theory only describes the classical gravitational field, and the final challenge will be to attain a quantum theory of gravity. Some progress in this direction has been made: for example, Feynman took the linearised theory of gravitational waves and showed that gravitational forces in this limit corresponded to the exchange of spin-2 **gravitons** – the spin reflecting the tensor nature of the field, just as vector fields are spin 1 and scalar fields are spin 0. But it is known that this linearised quantum theory is divergent at higher orders of perturbation theory, to an extent that cannot be made finite by **renormalization** in the same way that is successful in quantum electrodynamics. The reaction of most physicists is therefore that Einstein gravity can only be an **effective theory**, which is valid for large scales and low energies. An analogy would be the Fermi theory of the weak interaction; this is also not renormalizable, but becomes so because the nature of the theory changes at high energies where the existence of the W and Z bosons becomes apparent. Most probably something analogous will happen with gravitation, but after a century of effort we are still far from having definite ideas about the next step beyond Einstein's wonderful legacy.